JOSEPH LOUIS LAGRANGE,

Author of the Mécanique Analytique &c.

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THE

THE

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III. Solutions to the Questions in No. IX, and new Questions and Solutions for No. X, must arrive before the 15th of June next.

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THE

MATHEMATICAL DIARY.

N° IX.

BEING THE PRIZE NUMBER OF MR. EUGENIUS NULTY, PHILADELPHIA.

Dr. Bowditch's solution to the twenty-second question in No. VII. continued from the last number.

Again, by substituting in $d\varpi$, the value of dt, we get $d\varpi = \frac{\mathfrak{C}}{2r} \cdot \left\{ \frac{1}{r-z} + \frac{1}{rz.} \right\} \cdot \frac{hd\varphi}{\Delta c\varphi}$ substituting the value of z this may be reduced to

$$d\varpi = \frac{\mathfrak{C}h}{2r.(r-a)} \cdot \frac{d\varphi}{(1+n' \sin. \varphi^2)\,\Delta(c\varphi)} + \frac{\mathfrak{C}h}{2r.(r+a)}\,\frac{d\varphi}{(1+n \sin. \varphi^2)\,\Delta(c\varphi)}$$

whose integral depends on elliptical functions of the third species Π, so that we shall have

$$\varpi = \frac{\mathfrak{C}h}{2r(r-a)} \cdot \Pi(n', c, \varphi) + \frac{\mathfrak{C}h}{2r.(r+a)}\,\Pi(n, c, \varphi).$$

Which may be computed by any of the methods given by Le Gendre.

In a semi-vibration of the pendulum, φ changes from 0 to 90°, and if we suppose (ϖ) to be the value of ϖ, corresponding to $\varphi = 90^\circ$, we shall have

$$(\varpi) = \frac{\mathfrak{C}h}{2r.(r-a)} \cdot \Pi'(n'c) + \frac{\mathfrak{C}h}{2r.(r+a)} \cdot \Pi'(nc).$$

These definite functions of the *third* species may be reduced to those of the *first* and *second* species, by the formulas of Le Gendre, in vol. 1 of his work.

Then, in page 137, we have, by putting, $n' = \frac{(a-a')}{r-a} = \cot.\theta^2$,

$$\Pi'(n'c) = \frac{\sin.\theta \cdot \cos.\theta}{\Delta(b\theta)} \cdot \left\{ \frac{\pi}{2} + \frac{\sin.\theta}{\cos.\theta} \cdot \Delta(b\theta) \cdot \mathrm{F}'(c) + \mathrm{F}'(c) \cdot \mathrm{F}(b\theta) - \mathrm{F}'c \cdot \mathrm{E}(b\theta) - \mathrm{E}'(c) \cdot \mathrm{F}(b\theta). \right\}$$

And in p. 140, putting $n = -1 + b^2 \cdot \sin.\theta'^2 = -\frac{(a-a')}{r+a}$, we have

$$\Pi'(nc) = \mathrm{F}'(c) + \frac{\Delta(b\theta')}{b^2 \sin.\theta' \cdot \cos.\theta'} \cdot \left\{ \frac{\pi}{2} + \mathrm{F}'(c) \cdot \mathrm{F}(b\theta') - \mathrm{E}'(c) \cdot \mathrm{F}(b\theta') \right.$$

$-F'(c) . E(b\theta')\}$. Substituting these in (ϖ) and reducing we get by taking μ' so that $\tan. \mu' = \frac{rr + aa'}{a'\sqrt{(rr - aa)}}$

$$(\varpi) = \left\{ \sqrt{\left(\frac{(r^2 - a^2)(r^2 - a'^2)}{r^2(r^2 + a^2 + 2aa')}\right)} + E(b\mu') - F(b\mu')\right\} . F'(c) - F(b\mu')$$

$. E'(c)$; which being reduced to functions of the *first* and *second* species, may be very easily computed by means of tables I. and IX.

ARTICLE XVIII.

SOLUTIONS

TO THE QUESTIONS PROPOSED IN ARTICLE XVII. NO. VIII.

QUESTION I. (131.)—*By Mr. William Vogdes.*

Given $xy + \frac{y^3}{x} = 40$, and $\frac{x^3}{y} - xy = 96$, to find the values of x and y by a quadratic.

FIRST SOLUTION.—*By Mr. John B. Newman, jun. Phil.*

$x^2y + y^3 = 40x$, (A), from the first equation,
$x^3 - xy^2 = 96y$, (B), from the second;
$x^4 - y^4 = 3840$ (A×B), (H)
$x^3y + xy^3 = 40x^2$ (Ax), (C)
$x^3y - xy^3 = 96y^2$ (Bx), (D)
$2xy^3 = 40x^2 - 96y^2$ (C—D), (E)
$2x^3y = 40x^2 + 96y^2$ (C+D), (F)
$4x^4y^4 = 1600x^4 - 9216y^4$ (E×F), (G)
$x^4y^4 = 400x^4 - 2304y^4$ (G÷4), (I)
$1536000 = 400x^4 - 400y^4$, (H×400), (J)
$1904y^4 = 1536000 - x^4y^4$ (J—I), (K)
$3840y^4 = x^4y^4 - y^8$ (H×y^4), (L)
$1536000 - y^8 = 5744y^4$ (K—L)
$\therefore y^8 + 5744y^4 + 8248384 = 9784384$,
$y^4 + 2872 = 3128$; $\therefore y^4 = 256$, or $y = 4$;
Consequently $x^4 - 256 = 3840$, and $x = 8$.

SECOND SOLUTION.—*By Mr. Michael Floy, jun.*

Put $x = vy$, then the equations become

$$vy^2 + \frac{y^2}{v} = 40, \text{ and } v^3y - vy^2 = 96,$$

whence, $y^2 = \frac{40v}{v^2+1}$, and $y^2 = \frac{96}{v^3 - v}$;

hence, $\dfrac{40v}{v^2+1}=\dfrac{96}{v^3-v}$, $40v^4-40v^2=96v^2+96$,

$$\text{or } v^4-\frac{17}{5}v^2=\frac{12}{5};$$

whence, $v^2=\dfrac{17}{10}\pm\sqrt{\left(\dfrac{289}{100}+\dfrac{240}{100}\right)}=\dfrac{17}{16}\pm\dfrac{23}{10}=4$, or $-\dfrac{3}{5}$

Hence, by taking the positive value, $v=\pm 2$;

$$\text{and } y=\sqrt{\frac{40v}{v^2+1}}=\sqrt{16}=\pm 4.$$

Consequently, $x=vy=\pm 8$.

THIRD SOLUTION.—*By Mr. John Swinburn.*

Multiply the given equations crosswise, we have $96\left(xy+\dfrac{y^3}{x}\right)=40\left(\dfrac{x^3}{y}-xy\right)$: whence $x^4+\dfrac{17}{12}x^2y^2=\dfrac{5}{12}x^4$.

Completing the square, &c. $y^2=\frac{1}{4}x^2$; or $y=\frac{1}{2}x$: this substituted for y in either of the given equations, we obtain $x=8$; thence $y=\frac{1}{2}x=4$.

Or, the given equations may be expressed thus:

$$(x^2+y^2)\frac{y}{x}=40.$$

$$(x^2-y^2)\frac{x}{y}=96.$$

Put $x^2=mn$, and $y^2=\dfrac{m}{n}$: then by substitution,

$$\left(mn+\frac{m}{n}\right)\frac{1}{n}+40$$

$$\left(mn-\frac{m}{n}\right)n=96.$$

From the former of these, $n^2=\dfrac{m}{40-m}$: from the latter, $n^2=\dfrac{96+m}{m}$

$$\therefore \frac{m}{40-m}=\frac{96+m}{m}.$$

Whence, $m^2+28m=1920$.

Conseq. $m=32$.

And $n=\sqrt{\dfrac{m}{40-m}}=2$. Hence $x=\sqrt{mn}=8$ and $y=\sqrt{\dfrac{m}{n}}=4$.

QUESTION II. (132.)—*By Mr. William Vogdes.*

What is the area of a right angled triangle whose sides are in arithmetical proportion; the least side squared and divided by 6,

that quotient multiplied by the mean difference of the sides, will be equal to the area.

FIRST SOLUTION.—*By Mr. James Docharty, L. I.*

Put $x=$ the mean difference of the sides; then $3x$, $4x$, and $5x$, will represent the sides of the triangle, and $\frac{12x^2}{2}$ or $6x^2=$ the area: also, $\frac{9x^2}{6}\times x=\frac{3x^3}{2}=$ the area, by the question. Whence, $\frac{3x^3}{2}=6x^2$, or $3x^3=12x^2$; this divided by $3x^2$, gives $x=4$; $\therefore 12$, 15, and 20, are the sides of the triangle; and the area $=96$.

SECOND SOLUTION.—*By Mr. Augustus W. Smith.*

Let $x=$ the perp. $=$ the middle term of the series, and $y=$ the common difference; then $x-y=$ the base, and $x+y=$ the hyp. Now, (*Enc.* 47. I.) we have

$$x^2-2xy+y^2+x^2=x^2+2xy+y^2;\ \therefore\ x=4y:$$

and $\frac{yx^2-2xy^2+y^3}{6}=\frac{x^2-xy}{2}$, by the question.

Substituting the above value of x, in this equation, and reducing, we shall find, $y=4$; consequently, $x=16$, $x-y=12$, and $x+y=20$.

QUESTION III. (133.)—*By John Swinburne.*

Given the sum of two perpendiculars, drawn from a point in the side of an equilateral triangle upon the other two sides; to determine the triangle.

FIRST SOLUTION.—*By Mr. S. Hammond, Brooklyn.*

It is evident that the sum of the two perpendiculars, is equal to a perpendicular drawn from either angle to its opposite side. Hence we have given, one side of a right angled triangle, and the ratio of the two remaining sides, as 2 to 1; which is readily solved.

SECOND SOLUTION.—*By Hypathia, Penn.*

Let ABC (the fig. can be easily supplied) represent the required triangle; PE and PF the two perpendiculars, the sum of which is given; draw CD parallel to PE, and PG parallel to AB; then in the two similar triangles CGP and PFC, PC being common, the sides CG and PF are equal; wherefore CD ($=$CG$+$PE$=$ PF$+$PE) is given, which determines the triangle.

Cor. The point P is indeterminate, since PF$+$PE is always equal to CD.

Question IV. (134.)—*By Mr. John B. Moreau, N. Y.*

Five boys, A, B, C, D, and E, set out at the same time to run round a Park, the circumference of which is 500 yards. A goes 124 yards in a minute, B 119, C, 117, D 113, and E 107. I demand in what time will they come together again; and how often will A and E meet?

First Solution—*By Mr. George W. Taylor, Westchester.*

$124-107=17$, $124-113=11$, $124-117=7$, $124-119=5$; these remainders are the distances that A gains on the others in a minute. Then $17 : 500 :: 1$ min. : $\frac{500}{17}$ min. the time in which A would gain a round on E. In like manner it is found that A gains a round on D in $\frac{500}{11}$ min.; on C in $\frac{500}{7}$, and on B $\frac{500}{5}$ min.; and the least common multiple of these must be the time that will bring them all together again. Now it is evident, that 500 is that multiple; and that A will pass E 17 times, D 11 times, C 7 times, and B 5 times; also, A will walk round the Park 124 times, B 119, C 117, D 113, and E 107 times.

Second Solution.—*By Mr. Gerardus B. Docharty.*

Let v, w, x, y, and z, be the times run round the Park by A, B, C, D, and E, respectively; $t=$ the time when they will come together in minutes. Then, $\frac{124t}{500}=v$, or $t=\frac{500v}{124}=\frac{500w}{119}=\frac{500x}{117}=\frac{500y}{113}=\frac{500z}{107}$; and if we take v, w, x, y, and z, $=124$, 119, 117, 113, and 107, respectively, we shall have the same, 500 minutes $=8$ hours and 20 minutes; and A and E shall come together 17 times.

Question V. (135.)—*By Mr. John B. Moreau, N. Y.*

Given $\left\{\begin{array}{l} x(\sqrt{y}+1)+2\sqrt{xy}=55-y(\sqrt{x}+1) \\ \text{and } x\sqrt{y}+y\sqrt{x}=30; \end{array}\right\}$ to find the values of x and y.

First Solution.—*By Mr. Henry R. Lott, N. Y.*

By subtracting the second equation from the first, we have, $x+2\sqrt{xy}+y=25$; (A).$\therefore$extracting the square root, $\sqrt{x}+\sqrt{y}=\pm5$. Now, by dividing the second equation by the corresponding members of this, we get $\sqrt{xy}=6$ (B); substituting this value in equation (A) we shall find $x+y=13$; $\therefore x=13-y$; by substituting this value of x in equation (B) we get $\sqrt{(13y-y^2)}=6$; by squaring $13y-y^2=36$, or $y^2-13y=-36$; completing the square, &c.

$$y=4 \text{ or } 9, \text{ and } \therefore x=9 \text{ or } 4.$$

SECOND SOLUTION.—*By Mr. J. F. Jenkins.*

The first equation may be expressed thus, $x\sqrt{y}+x+2\sqrt{xy}+y\sqrt{x}+y=5$; from which subtract the second, and we have $x+2\sqrt{xy}+y=25$. Extracting the square root, $\sqrt{x}+\sqrt{y}=5$. Hence, $\sqrt{x}=5-\sqrt{y}$, and by squaring, $x=25-10\sqrt{y}+y$. Substitute this value for x in the second equation, which gives, after reduction, $y-\sqrt{y}=-6$; $\therefore$ completing the square, &c. $\sqrt{y}=3$ and $y=9$; whence x is readily found equal to 4.

QUESTION VI. (136.)—*By Mr. George Alsop, Philadelphia.*

In a given paraboloid, it is required to describe the greatest cylinder.

FIRST SOLUTION.—*By Mr. Silas Warner, B. C. Penn.*

Let $a=$ the height of the paraboloid, $x=$ the distance from the vertex to the top of the cylinder, and $p=$ the parameter: Then, $2\sqrt{px}=$ the diameter of the end, and $a-x=$ the height of the cylinder. Consequently, $(2\sqrt{px})^2\times(a-x)=$max., or $4px\times(a-x)=$ a max., $\therefore ax-x^2=$ a max.; by taking the fluxion, $adx-2xdx=0$; $\therefore a-2x=0$, or $x=\frac{a}{2}$.

SECOND SOLUTION.—*By Mr. Saxegotha Laws, Dover.*

Put a and b for the altitude and semidiameter of the given paraboloid, and $y=$ the semidiameter of the cylinder, and $x=$ the corresponding abscissa; then, $a-x=$ the height of the cylinder; $\therefore a:x::b^2:y^2=\frac{b^2x}{a}$. (*Cor.* 2, prop. 12, book 1, *Simp. Con.*) Put $p=3.14159$, &c. then $py^2=\frac{pb^2x}{a}=$ the area of the base of the cylinder; and $(a-x)\frac{pb^2x}{a}=$ a max.; which, by taking the differential and reducing, gives $x=\frac{a}{2}$; $\therefore y=\frac{b}{\sqrt{2}}$; $\therefore b\sqrt{2}=$ the diameter of the cylinder, $\frac{a}{2}=$ the height.

QUESTION VII. (137.)—*By Mr. Edward Giddings.*

Given the vertical angle, the ratio of the sides, and the radius of the inscribed circle, to construct the triangle.

FIRST SOLUTION.—*By Mr. Henry Doyle.*

Let $a=$ the given angle and abd be a triangle whose sides ab, ad, are in the given ratio. At the point p, which is easily found for $ap=\frac{r\times\cos.\frac{1}{2}a}{\sin.\frac{1}{2}a}$, $r=$ the given radius, erect the perpendicular

$pc = r$; then, as tangent to the circle *nps* and parallel to *bd*, draw *oq*, and *aoq* will be the required triangle, as is obvious from the construction.

SECOND SOLUTION.—*By Wm. H. Sidell, N. Y.*

With the given radius describe a circle; and take two of the radii, making an angle equal to the supplement of the given angle, at each of the points where they cut the circle draw a tangent, their intersection will form an angle equal to the given vertical one. The lengths of the tangents, from the point of intersection to the touching points, are easily found. If we now make one tangent in the given ratio to a part of the other measured from the vertex, from the centre erect a perpendicular on the line joining the terminating points, which produce the other way until it meets the circle, and at that point draw a third tangent, which continue both ways until it meets the other two produced, we will have the triangle required.

THIRD SOLUTION.—*By Edward Devoy, N. Y.*

Describe any triangle having its vertical angle equal to the given angle and the sides in the given ratio, in which inscribe a circle; produce those radii of this circle which pass through the points of contact, until they become equal to the radius of the given circle, the centre of which we will suppose to be situated at the centre of the first described circle, where these radii meet the given circle apply tangents. It is evident that these tangents are the sides of the required triangle.

QUESTION VIII. (138.)—*By Mr. Wm. Lenhart, York, Penn.*

If on a given base (286) of a plane triangle, between two acute angles, a semicircle be described, the circumference of which cuts the other two sides; and there be given the straight line joining the points of intersection 110, and the straight line bisecting the vertical angle and terminating in said line 68. It is required to determine the triangle.

FIRST SOLUTION.—*By Solomon Wright, Penn.*

Analysis. Let ACE be the triangle, (see fig. to solution quest. ix, page 154, *Math. Diary*, vol. 1); join OF, OD; then FAD ($=\frac{1}{2}$FOD) is given, and ADE is a right angle, because ADC, in a semicircle, is a right angle; $\therefore$ AED is given, and in the triangle FDE, the base FD, the vertical FED, and the line bisecting the vertical angle, are all given; which problem is constructed in *Simpson's Algebra*, prob. 72.

The composition and calculation are hence evident.

Second Solution.—*By John M. Wilt, Penn.*

The complement of $\frac{1}{2}$ the angle subtended by the chord (110) will be the angle at the vertex, and as 110 : 286 : : 68 : $176\frac{4}{5}$ = the line bisecting the vertical angle and terminating on the base.* Now, having the base, the vertical angle, and the bisecting line, we determine the triangle, (see prob. 72, Append. *Simpson's Alg.*) Vertical angle 67.23′, sides 162.7 and 306.

Third Solution.—*By Mr. Thomas J. Megear.*

It is shown in Lacroix's Geom. that the vertical angle of the triangle is measured by half the sum of the arcs, cut off the semicircle by the sides of the triangle; it is manifest that the required triangle is similar to one having a common vertical angle with it, and whose base is the given chord, (22.3) and (13.1) Now, the base of the latter triangle is to the given base of the required triangle, so is the line bisecting the vertical angle of the latter to that bisecting the vertical angle of the former; whence we have the base, the vertical angle, and the line bisecting it and terminated in the base, to construct the triangle; which is done in many books on Practical Geometry—*Simpson*, *Leslie*, &c.

Fourth Solution.—*By Mr. Wm. J. Lewis, N. Y.*

Let AEC (see fig. vol. 1, page 154), having the base AC=286, DF=110, and the line bisecting the vertical angle FED and terminating in DF=68, from the centre O draw the radii OF, OD. Now all the sides of the triangle OFD being given, the angle FOD is also given, and is double the angle FCD (*Euc.* 20,3 ,) and CFE being a right angle, the vertical angle AEC is also given. Hence, in the triangle EFD, we have the base FD, the vertical angle FED, and the line bisecting the vertical angle and terminating in the base FD. Now we may consider the triangle as constructed by prob. 72, *Simpson's Algebra.* Upon FD construct the isosceles triangle FOD, having each of its sides equal to half of AC; from O, as a centre, describe the semicircle AFDC, and produce EF, ED to cut the circle in the points A and C, join AC, and AEC will evidently be the triangle required.

Calculation. Join FC and DA. Now in the right angled triangles EFC and EDA, we have the angle AED common to both, and the sides EF and ED (as calculated by prob. 72, *Simpson's Alg.* App.) to find the sides EC and EA, by Trigonometry.

Question IX. (139.)—*By Mr. John Delafield, jun.*

Being called upon to survey a rectangular piece of ground, whose length I found to be double the breadth, and that the length of a line, in feet, drawn from the top of a perpendicular pole, placed in one of the corners to the opposite corner, (whose height

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* For demontration see Diary, No. 6. Sol. 3. Quest. 9.

was equal to one fourth of the breadth) was equal to the area in square perches; to determine the sides and area of the rectangle.

FIRST SOLUTION.—*By Mr. George Evans, N. Y.*

Let x represent the height of the pole; then $4x =$ the breadth of the rectangle, and $8x =$ the length. Consequently the length of the line, in perches, is $\sqrt{(8x^2+4x^2)+x^2}=\sqrt{81x^2}=9x$, and the area of the rectangle $=8x\times 4x=32x^2$; $\therefore\ 32x^2=16\frac{1}{2}\times 9x$, and $x=4.640625$. Hence, $8x=37.125=$ the length, $4x=18.5625=$ the breadth, and $37.125\times 18.5625=689.1328125=$ the area of the rectangle.

SECOND SOLUTION.—*By William Vogdes.*

Let $x =$ the breadth of rectangle, and $2x =$ the length of do.; then, $\sqrt{5x^2} =$ diagonal; $\frac{x}{4} =$ height of the pole; $2x^2 =$ area of rectangle. Whence,

$$\sqrt{\left(\frac{x^2}{16}+5x^2\right)}=\sqrt{\frac{81x^2}{16}}=\frac{9x}{4}= \text{length of line in perches.}$$

Consequently, $\frac{9x}{4}\times 16\frac{1}{2}=2x^2$.

$$8x=148.5 \text{ or } x=18.5625 = \text{breadth}$$
$$2x=37.125 = \text{length}$$
$$2x^2=689.1328125 = \text{area.}$$

QUESTION X. (140.)—*By Diophantus.*

To find two numbers whose sum shall be a cube, and the sum of their squares, increased by thrice their sum, shall be a square.

FIRST SOLUTION.—*By John D. Williams, Dighton, Mass.*

Let x and $1-x$ represent the numbers, and the first condition is satisfied; then, by the question, $(1-x)^2+x^2+3(1-x+x) =$ a square, or $1-2x+2x^2+3 =$ a square; assume its side $=2-ax$; then $1-2x+2x^2+3=(2-ax)^2=4-4a+xa^2x^2$; whence, by reduction, we get $x=\frac{4a-2}{a^2-3}$. Where a may be any number greater than 2, let $a=5$; then $x=\frac{18}{23}$, and $1-x=\frac{5}{23}$.

SECOND SOLUTION.—*By Mr. James Divver, S. C. College.*

Let x and y represent the numbers; then $x+y =$ a cube; and $x^2+3(x+y)+y^2 =$ a square; assume its $= x+y$; then $x^2=3(x+y)+y^2=x^2+2xy+y^2$; therefore, $3(x+y)=2xy$. Now if we take $x+y=(2)^3=8$; we shall find $x=2$, and $y=6$, numbers which answer the conditions of the question.

THIRD SOLUTION.—*By Mr. Nathan Brown.*

Let the numbers be rx and sx. Then $rx+sx=x^3 \therefore r+s=x^2$. And $r^2x^2+s^2x^2+3(r+s)x=\square$. Put $3(r+s)x=2rsx^2 \therefore 2rsx=3(r+s)=3x^2 \therefore 3x^2 \therefore 3x=2rs \therefore 9x^2=4r^2s^2=9(r+s) \therefore$ by quadratics $s=\frac{9+\sqrt{(16r^2+9)}}{8r^2}$. It now remains to make $16r^3+9$ a square, which is manifestly the case when $r=1$. Hence the numbers are 6 and 2.

QUESTION XI. (141.)—*By Diophantus.*

To find two numbers, whose sum shall be an integral cube, and such that the square of each increased by the other shall be an integral square.

FIRST SOLUTION.—*By Theodore Strong, Professor of Mathematics, Rutgers College.*

The two last conditions of this question cannot be answered in integers, as required, which may be shown as follows:

Firstly, suppose the numbers required are fractions; then they must be of the form $p-\frac{m^2}{n^2}, \frac{m}{n}$, m and n being integers, such that m and n are prime to each other, and p an integer; these give $\frac{m^2}{n^2}+p-\frac{m^2}{n^2}=p$ an integer; but $\left(p-\frac{m^2}{n^2}\right)^2+\frac{m}{n}=$ an integer, or $p^2-2p\cdot\frac{m^2}{n^2}+\frac{m^4}{n^4}+\frac{m}{n}=$ an integer; $\therefore$ multiply by the integer n^2, and the product will be an integer; therefore, $p^2n^2-2pm^2+\frac{m^4}{n^2}+mn=$ integer; but $p^2n^2-2pm^2+mn$ is an integer; $\therefore \frac{m^4}{n^2}=$ an integer, or $\frac{m^2}{n}=$ integer; hence, m is divisible by n, against the hypothesis.

Secondly, the numbers cannot be integers, for let x and y denote them; then $x^2+y=(\text{integer})^2=(x+z)^2=x^2+2xz+z^2$; $\therefore y=2xz+z^2$, and consequently y is greater than x, since z is an integer. Again, $y^2+x=(\text{integer})^2$, per question; therefore, by the argument just used, x is greater than y; hence the squares cannot be integers, unless x is greater than y and y greater than x at the same time, which is impossible. I shall, therefore, solve the question with the omission of the integral condition in the two last conditions. Assume a^6-y, and y for the numbers, then, per question, $(a^6-y)^2+y=(a^6-y+b)^2$, (by assumption,) $\therefore y=\frac{2a^6b+b^2}{2b+1}$, but $y^2+a^6-y=\left(\frac{2a^6b+b^2}{2b+1}\right)^2+\frac{a^6-b^2}{2b+1}=$

$\frac{(2a^6+b^2)^2+(a\ -b^2)\times(2b+1)}{(2b+1)^2}=$ a square; $\therefore$ $(2a^6b+b^2)^2+(a^6-b^2)\times(2b+1)=a^6+2ba^6+(4a^{12}-1)\times b^2+(4a^6-2)b^3+b^4$ $=$ a square $=(a^3+pb+b^2)$, (by assumption). Assume $p=a^3$, then I have $4a^{12}-1+(4a^6-2)b=a^6+2a^3+2a^3b$; $\therefore$ b $=-\frac{2a^6+a^3+1}{2}$, this gives $y=\frac{2a^9-(a^6+1)}{4a^3}$, and $a^6-y=$ $\frac{2a^9+a^6+1}{4a^3}$, in which any integer greater than 1, may be assumed for a. Let $a=2$, then $y=\frac{959}{32}$ and $a^6-y=\frac{1089}{32}$, which numbers will be found to answer.

Otherwise; let $1-y$, and y denote the numbers, then $(1-y)^2+y=1-y+y^2$, and $y^2+(1-y)=1-y+y^2$; $\therefore$ the question will be answered by making, $1-y+y^2=$ sq. $=(y+a)^2$, (by assumption) $\therefore$ $1-y=2ay+a^2$, and hence $y=\frac{1-a^2}{2a+1}$; let $a=\frac{1}{2}$, then $y=\frac{3}{8}$ and $1-y=\frac{5}{8}$, which numbers will be found to answer, but the two squares are equal in the hypothesis here made.

Second Solution.—*By Omicron, S. C.*

Let $x=$ one of the numbers, s^3 their sum, then $s^3-x=$ the other, and per question $x^2-x+s^3=\square$, and $s^6-2s^3x+x^2+x=\square$. Make a side of this last equation $=s^3-x+\frac{1}{2}$, and we shall have $x=\frac{4s^3+1}{8}$; then substituting this value of x in the first formula and reducing it we obtain $16s^6+40s^3-7=\square$, which condition is answered by assuming $s=1$, in which case $\frac{5}{8}$ and $\frac{3}{8}$ are numbers answering the required conditions.

Third Solution.—*By Mr. J. Ingersoll Bowditch.*

Let the numbers be $4x^3+y$ and $4x^3-y$; $\therefore$ $8x^3=$ a cube, per question $(4x^3+y)^2+4x^3-y=\square$, and $(4x^3-y)^2+4x^3+y=\square$. If we make $8x^3=1$; $\therefore$ $x=\frac{1}{2}$, consequently by substituting this gives,

$(\frac{1}{2}+y)^2+\frac{1}{2}-y=\square$, and $(\frac{1}{2}-y)^2+\frac{1}{2}+y=\square$, or $\frac{3}{4}+y^2=\square$, and $\frac{3}{4}+y^2=\square$. Now make $\frac{3}{4}+y^2=(y+n)^2$; $\therefore$ $y=\frac{3}{8n}-\frac{n}{2}$.

As x is $\frac{1}{2}$, y must be less than $\frac{1}{2}$ and positive. We see that n must be greater than $\frac{1}{2}$ and smaller than $\frac{1}{2}\sqrt{3}$, or the limit of the greatest value of n is between $\frac{7}{8}$ and $\frac{13}{16}$.

Put, for example, $n=\frac{3}{4}$; $\therefore$ $y=\frac{1}{8}$ and $x=\frac{4}{8}$; the numbers are $\frac{5}{8}$ and $\frac{3}{8}$, which answer the conditions of the question, when the last condition is read, " and such that the square of each increased by the other shall be a square.

QUESTION XII. (142.)—*By Philo. Winchester, Va.*

Required the dimensions of a right angled triangle, the hypothenuse of which is given; the perpendicular being added to twice the base being a maximum.

FIRST SOLUTION.—*By Mr. Marcus Catlin.*

Let $a =$ the hypothenuse, $x =$ the base; then $(a^2-x^2)^{\frac{1}{2}}=$ the perpendicular. Therefore, $\sqrt{(a^2-x^2)}+2x =$ a maximum; $\therefore$ $(a^2-x^2)^{-\frac{1}{2}}xdx+2dx=0$, or $(a^2-x^2)^{-\frac{1}{2}}\times x+2=0$; $\therefore$ $5x^2=4a^2$; hence, $x=\sqrt{\frac{4}{5}a^2}=2a\sqrt{\frac{1}{5}}$, and $(a^2-x^2)^{\frac{1}{2}}=(a^2-\frac{4}{5}a^2)^{\frac{1}{2}}=\sqrt{\frac{1}{5}a^2}=a\sqrt{\frac{1}{5}}$. Consequently, $\frac{1}{5}a^2=$ the area.

Cor. In a similar manner, it is shown that if the perpendicular added to n times the base be a maximum, the base will be $=\frac{na}{\sqrt{(n^2+1)}}$, and the perpendicular $=\frac{a}{\sqrt{(n^2+1)}}$; that is, the base $= n$ times the perpendicular. Hence the area $=\frac{na^2}{2n^2+2}$. If $n=1$, the base and perpendicular must be equal, and the $=\frac{1}{4}a^2$.

SECOND SOLUTION.—*By Mr. Benjamin Hallowell, Alex.*

This question is precisely the same as question 10, No. 3, of the Diary; and as four fluxional solutions were published to that, I shall send a geometrical one.

Construction. On the given hypothenuse AB, describe the semicircle ACB, and erect the perpendicular BF=2AB. Draw ACF and join BC; then ACB will be the required triangle; for, draw any other triangle AEB and produce AE making EG = 2EB. Join GB, and on it produced let fall the perpendicular AH. Since FB=2AB, CF=2CB; and since GE=2EB, GH=2AH; wherefore the two triangles FBA and GHA are similar. But AB is greater than AH, because the hypothenuse of a right angled triangle is greater than the leg; therefore, AF, or AC+2CB, is greater than AG, or AE+2EB. Q.E.D.

Cor. BC=2AC.

QUESTION XIII. (143.)—*By Mr. Silas Warner.*

It is required to inscribe in a given hyperbola, a rectangle whose area shall be a maximum.

FIRST SOLUTION.—*By Mr John F. James, Phi.*

The equation of a hyperbola is $a^2y^2=c^2(ax+x^2)$ in which a and c are the axes, x and y the abscissa and ordinate. Let d be the altitude of the given hyperbola; one side of the required rectangle is $2y=2\sqrt{\frac{c^2(ax+x^2)}{a^2}}$ and by ques. $(d-x)\cdot\frac{c}{a}\cdot 2\sqrt{ax+x^2}$

$=$ max. or $(d-x) \cdot \sqrt{ax+x^2}=$max. The fluxion is $(d-x) \times \frac{a\dot{x}+2x\dot{x}}{2\sqrt{ax+x^2}} - \dot{x} \times \sqrt{ax+x^2} = 0$, reducing, &c. we get $x^2 + \frac{3a-2d}{4}x = \frac{ad}{4}$, whence we obtain $x = \sqrt{\left(\frac{ad}{4}+\frac{(3a-2d)^2}{64}\right)} - \frac{3a-2d}{8}$. Or the same thus: let a be the semi-transverse axis, x the abscissa, and d the altitude of the hyperbola. Then the sub-tangent of the hyper, is $\frac{2ax+x^2}{a+x}$; by Simpson's Fluxions, art. 429, $\frac{2ax+x^2}{a+x} = d-x$ when the rectangle is a maximum, that is $x^2 + \frac{3a-d}{2}x = \frac{ad}{2}$, whence $x = \sqrt{\left(\frac{ad}{2}+\frac{(3a-d)^2}{16}\right)} - \frac{3a-d}{4}$.

SECOND SOLUTION.—*By Henry Dose, Esq. Natchez, member of the bar of Louisiana, late of West-Point.*

Let the equation of the hyperbola be $b^2x^2-a^2y^2=a^2b^2$ and the given abscissa$=c$. Then we have $2y(c+x)=$max.; and therefore $\frac{dy}{dx}=\frac{b^2x}{a^2y}=\frac{y}{c-x}$. Eliminating y there results $x^2-\frac{c}{2}x=a^2$, an equation which is easily constructed.

THIRD SOLUTION.—*By Omicron, N. C.*

Let $a =$ semiaxis major, $b =$ semiaxis minor, $c =$ distance from the centre of the hyperbola to the extremity of the given ordinate, and $x =$ distance from the centre of the hyperbola to the extremity of the ordinate which forms half the breadth of the rectangle; then per conics, $a^2 : b^2 :: x^2-a^2 : \frac{a^2}{b^2}(x^2-a^2) =$ the square of the semi-breadth of the rectangle; consequently, $(c-x) \cdot (x^2-a^2)^{\frac{1}{2}}$, or $c-x)^2 \cdot (x^2-a^2)$ is a maximum, the differential of which is $-2dx(c-x) \cdot (x^2-a^2)+2xdx(c-x)^2=0$, whence $x=\frac{2}{4}\pm\frac{1}{4}\sqrt{8a^2+c^2}$.

QUESTION XIV. (144.)—*By Mr. William J. Lewis.*

A circle and its tangent being given in position and magnitude, it is required to draw a line from the extremity of the tangent cutting the circle so that the part of the line intercepted by the circle shall be equal to the tangent.

FIRST SOLUTION.—*By Mr. Jacob Barton.*

Let ABC be the circle whose centre is 0, and AT its tangent,

given in position and magnitude. With the centre A and distance AT describe an arc cutting the circle in B; join AB: draw OD, bisecting AB in D, and with the centre O and radius OD describe the circle DEF. From the extremity T of the given tangent, draw a tangent to the concentric circle, and the part cut off by the given circle will be equal to the tangent AT. The demonstration is evident from the construction.

SECOND SOLUTION.—*By Mr. Enoch Lanning.*

This is a particular case of an old but easy problem, and may be thus done: find the side of a square equal to the difference of the squares of the radius of the circle and half the tangent; with this line, as a radius and concentric with the given circle, describe another; from the extremity of the tangent draw tangents to the interior circle, and the portions of those tangents, which become chords in the outer circle, are respectively equal to the given tangent. This needs no demonstration.

THIRD SOLUTION.—*By Mr. James Sloane, Middleton Academy, New-Jersey.*

This question does not admit of a solution, except when the given tangent is equal to or less than the diameter of the given circle. *Construction.* When the given tangent is equal to the diameter, draw a line through the centre of the circle from the extremity of the tangent.

But if the tangent be less than the diameter, it is evident that the distance of the part intercepted by the circle from its centre is equal to the cosine of half the arc cut off by the chord. Therefore, from the extremity of the tangent draw a tangent line to the concentric circle described with the cosine (as radius) of half the arc cut off by the given tangent as a chord. The demonstration is evident.

QUESTION XV. (145.)—*By Mr. Nathan Brown.*

The angular points of an acute angled triangle are the centres of three circles, each of which cuts off half the area of the triangle. Given the radii of these circles to determine the triangle, whose angular points are made by the intersection of the circles with one another.

FIRST SOLUTION.—*By Mr. John D. Williams.*

Let ϕ, ϕ', ϕ'', denote the three angles of the triangles, and r, r', r'', the corresponding given radii; $p =$ the semi-circumference radius (1). Then $\phi+\phi'+\phi''=p$, and $r^2\phi=r'^2\phi'=r''^2\phi''$, per quest. $\therefore\ \phi'=\frac{r^2\phi}{r'^2}$, and $\phi''=\frac{r'^2\phi'}{r''^2}$; substitute these and reduce, and we

have $\phi = \frac{r'^2 r''^2 p}{r'^2 r''^2 + r^2 r'^2 + r^2 r''^2}$, and the degrees in $\phi =$ $\frac{180 \times r'^2 r''^2}{r'^2 r''^2 + r^2 r'^2 + r^2 r''^2}$. Also, the degrees in $\phi' = \frac{180 \times r^2 r''^2}{r'^2 r''^2 + r^2 r'^2 + r^2 r''^2}$; whence the triangle is given in species, and the area is also given, for it is $= r^2 \phi = \frac{r^2 r'^2 r''^2 p}{r'^2 r''^2 + r^2 r'^2 + r^2 r''^2}$. Hence the area and angles being known, the sides are easily found, and thence every thing else.*

Second Solution.—*By Doctor Bowditch.*

Let A, B, C, be taken for the angles of the proposed acute angled triangle, and a, b, c, for the radii of the proposed circles. Then the area of the sectors of circles included by these angles, and the corresponding radii being equal to each other, gives $Aa^2 = Bb^2 = Cc^2$, and as $A + B + C = 180°$, if, for brevity, we put $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{e^2}$, these give

$$A = 180° . \frac{e^2}{a^2}, \quad B = 180° . \frac{e^2}{b^2}, \quad C = 180° . \frac{e^2}{c^2}.$$

Hence we have the three angles of the proposed triangle, and therefore the ratio of their sides. Assuming now any triangle of this species, we may compute its area, and then the radius of the circular sector described between the angle which includes half that area. Then, as this radius is to the given radius, so is any side of the assumed triangle to the corresponding side of the required triangle. By these means the sides of the triangle may be found; and then describing about A, B, C, the arches of the proposed circles, we may obtain the angular points of the triangle found by the intersection of these circles.

Otherwise. We may observe that if r be the radius of the circle circumscribing the proposed acute angled triangle, the sides of the triangle will be respectively $2r$. sin. A, $2r$ sin. B, $2r$ sin. C, and the half area of the triangle r^2. sin. A, sin. B, sin. C; but, by question, this area is $= \frac{A}{360°} \pi a^2$, equalling these two expressions we get the following expression of r^2, which gives r after having found, as before, A, B, C,

$$r^2 = \frac{\frac{A}{360°} . \pi a^2}{\text{sin. A, sin. B, sin. C.}}$$

From this value of r we get immediately the required sides, $2r$ sin. A, $2r$ sin. B, $2r$ sin. C.

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* Professor Strong's solution was exactly like this.

THIRD SOLUTION.—*By J. Ingersoll Bowditch.*

Let $a=4$, $b=5$, and $c=6$, be the radii of the circles. Hence $LL=\frac{2c\pi C}{360}$ $MM=\frac{2a\pi A}{360}$ $NN=\frac{2b\pi(180-A-C)}{360}$.

Then the area of the sector $CLL=\frac{c^2\pi C}{360}$

$$AMM=\frac{a^2\pi A}{360}$$

$$BNN=\frac{b^2\pi(180^\circ-A-C)}{360}$$

Per question, $c^2\pi C=a^2\pi A \therefore C=\frac{a^2}{c^2}A$

and $c^2\pi C=b^2\pi(180-A-\frac{a^2}{c^2}A)$

Consequently $A=\frac{b^2c^2180^\circ}{a^2c^2+b^2c^2+b^2a^2}$, $C=\frac{b^2a^2180^\circ}{a^2c^2+b^2c^2+b^2a^2}$, and

$B=\frac{180a^2c^2}{a^2c^2+b^2c^2+b^2a^2}$, put $a=4$, $b=5$, $c=6$, $\therefore$ $A=86^\circ 21\frac{1}{4}$, $B=55^\circ 16$, $B=38^\circ 22\frac{3}{4}$.

Having now the three angles and the area of the triangle, we may construct the triangle as follows:

Make x the base, then sin. $A : x ::$ sin. $C : AB \therefore AB=\frac{\text{sin. } C}{\text{sin. } A}$ $x=mx$, and $AC=\frac{\text{sin. } B}{\text{sin. } A}$ $x=nx$. The area of a triangle whose sides are x, mx, nx, is $\frac{1}{4}\sqrt{4x^4m^2-(x^2+m^2n^2-n^2x^2)^2}=\frac{2c^2\pi C}{360^\circ}\therefore$

$\frac{x^2}{4}\sqrt{4m^2-(1+m^2-n^2)^2}=\frac{2c^2\pi C}{360}$, or $x^2\sqrt{4m^2-(1+m^2-n^2)^2}=\frac{4}{5}\pi C$, putting for m, n, and π, their values .62213, .82348, and 3.14159, we obtain $x^2=94.23 \therefore x=9.78$. Hence $mx=6.084$ and $nx=8.054$.

Having now the sides, construct the triangle, and with the given radii draw the circles and the points of bisection will be the angular points of the required triangle.

FOURTH SOLUTION.—*By Mr. James Macully, N. Y.*

The required triangle may be easily determined when the acute angled triangle, mentioned in the question, is known. Let, therefore, the angles of this triangle be a, b, c; their opposite sides a', b', c'; and the given radii of the circles described at their vertices r, r', r''. We shall then have $a+b+c=180^\circ$; $ar^2=br'^2=cr''^2$; and $a'b'.\text{sin. } c=a'c'.\text{sin. } b=b'c'.\text{sin. } a=2ar^2$; from the first

three of which we easily find the angles a, b, c, and the remaining equations will give the sides a', b', c'.

Question XVI. (146.)—*By Mr. Thomas J. Megear.*

On a given base it is required to construct a right angled triangle, such that a perpendicular drawn from the right angle to the hypothenuse shall cut off the greater segment of it equal to the remaining side of the triangle.

First Solution.—*By John C. Reid.*

Let $a =$ the base, and $x =$ the perpendicular; then $\frac{a^2}{x}=$the hypothenuse, per question, and $\therefore$ (E.47.1), $a^2+x^2=\frac{a^4}{x^2}$, or $a^2x^2+x^4=a^4$; $\therefore$ completing the square,

$$x^4+a^2x^2+\frac{a^4}{4}=\frac{a^4}{4}+a^4=\frac{5a^4}{4};$$

$\therefore$ extracting the root, $x^2+\frac{a^2}{2}=\pm\frac{a^2}{2}\sqrt{5}$;

$$\text{and consequently, } x=\pm\sqrt{\pm\frac{a^2}{2}\sqrt{5}-\frac{a^2}{2}},$$

$$\text{or } x=\pm\frac{a}{2}\sqrt{\pm 2(\sqrt{5}-1)}.$$

Second Solution.—*By Mr. Wm. Lenhart.*

On AB, the given base of the triangle, describe the semicircle ACB (the figure can be readily supplied), and let AB be divided in D in extreme and mean ratio. Draw DC perpendicular to AC which produce to meet the perpendicular BE in E, so shall ABE be the triangle required.

Demonstration. Join BC, and since $AB\times BD=\overline{BC}^2=\overline{AD}^2$ by construction, we have BC=AD; and by reason of the similar triangles ACD, BEC, we have BE : BC : : AC : AD, whence AC =BE. Which was to be done.

Question XVII. (147.)—*By Mr. James Macully.*

It is required to inscribe in a given paraboloid the greatest possible cone, the diameter of whose base shall be perpendicular to the base of the paraboloid.

First Solution.—*By Mr. Eugenius Nulty.*

Let r and p be the radii of the bases of the given solid and the required cone, and $2a$ the parameter of the generating parabola.

The equation of the surface of the paraboloid is $a^2+y^2+2az=r^2$, and at the point (yz) in which the base of the cone touches this surface $y^2=p^2-a^2$ and $z=p+a$. We have therefore $(a+p)^2=r^2-x^2$ and $p^2(r+x)=$ a max. These equations give $\frac{x}{a+p}=\frac{p}{2(r+x)}$; and eliminating p, then results $(2x-r)^2 \cdot (r+x)=a^2(r-x)$, a cubic equation, from which $x=m$, and therefore $p=\sqrt{(r^2-m^2)}-a$

If the greatest right cone be required, we shall have $\frac{x}{a+p}=\frac{p}{2x}$ and $(r^2-2x^2)^2=a^2(r^2-x^2)$, from which x may be found by quadratics.

Second Solution.—*By Professor Strong.*

It is evident that the vertex of the cone must be at the extremity of one of the diameters of the base of the paraboloid, and that the base of the cone is the inscribed circle of a parabola having its plane perpendicular to the same diameter, its parameter being the same with that of the generating parabola of the given paraboloid. Let then $r=$ the radius of the base of the paraboloid, and $x=$ the height of the cone, and R the radius of the cone's base; then we have $R=\frac{2\sqrt{(2rx-x^2)}-p}{2}$; hence I have $\left(2\sqrt{(2rx-x^2)}-p\right)\sqrt{x}=$ a max. differentiate and reduce, and there results the equation $8rx-6x^2=p\sqrt{(2rx-x^2)}$, ($p=$ the parameter.) This equation can be readily reduced to a cubic, which solved gives x, and thence the cone is determined.

Question XVIII. (148.)—*By Omicron, N. C.*

Given the difference of the lengths of the shadow of a given object on two given days at noon, to determine the latitude.

First Solution.—*By Doctor Bowditch.*

Let the lengths of the shadow be represented in parts of the length of the stile taken as unity. The difference of these lengths being a. Let L be the latitude, D, d, the declinations; then we shall have

$$a=\text{tang.}\,(L-d)-\text{tang.}\,(L-D)=\frac{\sin.\,(D-d)}{\cos.\,(L-d)\,.\,\cos.\,(L-D)}$$

Whence we get $2\cos.\,(L-d)\,.\,\cos.\,(L-D)=\frac{2(D-d)}{a}$, or by reduction, $\cos.\,(L-d-D)=-\cos.\,(D-d)+\frac{2\sin.\,(D-d)}{a}$ the second member being known, may be put under the form $\pm b$, and then we shall have $L=\frac{\pm b+d+D}{2}$.

SECOND SOLUTION.—*By a Correspondent, Lexington, Ken.*

Let a = the difference of the lengths of the shadow; b = the perpendicular height of the object from which the shadow is projected; d and d' = the Sun's declination on the given days; and x = the required latitude. The complements of the Sun's meridian altitude on the two given days will be $d \pm x$ and $d' \pm x$, respectively; and hence, by the question, we soon obtain $\frac{a}{b}$ = tang. $(d \pm x)$ — tang. $(d' \pm x)$; from which x is readily found.

QUESTION XIX. (149.)—*By M. O'Shannessy, A. M.*

Required to find the content of the least sphere that shall touch any three right lines given in space.

FIRST SOLUTION.—*By Mr. J. Ingersoll Bowditch.*

Let us, for simplicity, take the first line for the axis of x, and we shall have for equations of the second and third lines,

$$z''=a''x''+\alpha'' \quad (1) \qquad z'''=a'''x'''+\alpha''' \quad (3)$$
$$y''=b''x''+\beta'' \quad (2) \qquad y'''=b'''x'''+\beta''' \quad (4)$$

The radius of the sphere will be in the planes drawn perpendicular to these lines through the points where the sphere touches these lines, and the intersections of these planes will give the centre. We have for equations of these planes

$$x=x'$$
$$(x-x'')+b''(y-y'')+a''(z-z'')=0 \quad (5)$$
$$(x-x''')+b'''(y-y''')+a'''(z-z''')=0 \quad (6)$$

The co-ordinates of the centre of the sphere being x, y, and z, the radius will be represented by the three following expressions, denoting the distances of that centre from these lines, namely,

$$\sqrt{y^2+z^2} \qquad \sqrt{(x-x'')^2+(y-y'')^2+(z-z'')^2}$$
$$\sqrt{(x-x''')^2+(y-y''')^2+(z-z''')^2}$$

which formulas give the two equations,

$$z^2+y^2=(x-x'')^2+(y-y'')^2+(z-z'')^2 \quad (7)$$
$$z^2+y^2=(x-x''')^2+(y-y''')^2+(z-z''')^2 \quad (8)$$

Substitute in equation (5) the values z'' and y'' and it becomes $(x-x'')+b''(y-b''x''-\beta'')+a''(z-a''x''-\alpha'')=0$; put for brevity $1+a''a''+b''b''=m''$; $a''\alpha''+b''\beta''=n''$, and we have

$$x+b''y+a''z=m''x''+n'' \quad (5')$$

Equation (7) developed becomes $x^2-2xx''+x''x''+y''^2-2y''y+z''^2-2z''z=0$; substituting, as above, and making $\alpha''^2+\beta''^2=p''$, we have

$$-2x''(a''z+x+b''y-n'')+m''x''^2-2\beta''y-2\alpha''z+x^2+p''=0;$$

multiply this by m'' and we have $-2x''m''(a''z+x+b''y-n'')+m''x''^2-2\beta''m''y-2m''\alpha''z+m''x^2+m''p''=0$; putting for $m''x''$ its value in equation (5′) we get, by changing the signs,

$$(x+a''z+b''y-n'')^2+2m''\beta''y+2\alpha''m''z-m''x^2-m''p''=0 \quad (9).$$

By a similar procedure, with the equations (3), (4), (6), (8), we obtain, $(x+a'''z+b'''y-n''')^2+2m'''\beta'''y+2\alpha'''m'''z-m'''x^2-m'''p'''=0$ (10).

Having thus two equations of the second degree, we may find x from each of these in terms of yz. Equalling these two values, we get a final equation between yz, which, for brevity, we shall put as follows : $F(yz)=0$ (11)

$F(yz)$ denoting a function of yz, which is easily found, but would be rather complex to print in this work.

The radius of the sphere being $\sqrt{z^2+y^2}$, we have, per question, $z^2+y^2=\text{minimum} \therefore d(z^2+y^2)=0 \therefore zdz+ydy=0 \therefore \frac{dz}{dy}=-\frac{y}{z}$;

substituting the value of $\frac{dz}{dy}$ in the differential of equation (11) we get another equation of yz, with which, and equation (11), the values of zy may be found.

Let us suppose a simple case, for example, that the second line be in the plane of xy and parallel to the first; in which case we have $z''=o, a''b''\alpha''=o, y''=\beta'', 1+a''a''+b''b''=m''=1, a''\alpha''+b''\beta''=n''=o, p''=\beta''$. Hence equation(9)becomes $x^2+2\beta''y-x^2-\beta''^2=o \therefore 2\beta''y-\beta''^2=o \therefore 2y=\beta'' \therefore y=\frac{1}{2}\beta''$. Substituting this in (10) we get

$$(x+a'''z+\frac{\beta''b'''}{2}-n''')^2+2\alpha'''m'''z-m'''x^2+m'''\beta'''\beta''-m'''p'''=o$$

(10'). Let the third line be drawn midway between the two parallel lines through the axis of y. The equations of this line will become

$$z'''=a'''x''' \quad y'''=\beta''' \quad \text{But } y'''=\tfrac{1}{2}y'' \therefore \beta'''=\tfrac{1}{2}\beta''$$

and $b'''=o$, $\alpha''=o, n'''=o, m'''=1+a'''^2$, $p'''=\beta'''^2=\beta\frac{1}{4}\beta''^2$.

Therefore (10') becomes $(x+a'''2)^2-x^2(1+a'''^2)+(1+a'''^2)\frac{\beta''^2}{4}$

$=o$ which developed gives $a'''^2(x^2-z^2)-2a'''^2xz-(1+a'''^2)\frac{\beta''^2}{4}=o$.

The radius of the sphere is $\sqrt{z^2+y^2} \therefore z^2+y^2=$ minimum or $z^2+\frac{1}{4}\beta''^2=$ minimum. Taking its differential we have $2zdz=o \therefore z=o$. Hence $a'''^2x^2=(1+''a^2)\frac{\beta''^2}{4} \therefore x=\frac{\beta''^2}{2a'''}\sqrt{1+a^{2'''}}$.

The radius of the sphere is $x \times$ by sin. angle whose tangent is a'' call the angle $v \therefore r=x \sin. v$. But $\sqrt{1+a'''^2}=\frac{1}{\cos. v}$

$\therefore x=\frac{\beta''}{2a''' \cos. v}$. Hence $r=\frac{\beta'' \sin. v}{2a''' \cos. v}=\frac{\beta''}{za'''}\tan. v=\frac{1}{2}\beta''$.

Then $x=\frac{\beta''}{2a''' \cos. v}$, $y=\frac{1}{2}\beta''$, and $z=o$.

SECOND SOLUTION.—*By Professor Strong.*

For brevity, I use the construction and notation given by Dr. Adrain in the 7th No. of the Diary, (see the solution given by him of the 20th question of No. 6, No. 7), where I have

$$(a+x)^2+(z \cos. a-y \sin. a)^2=r^2$$
$$(a-x)^2+(z \cos. a+y \sin. a)^2=r^2$$
$$bx=yz. \qquad (1)$$

Add the two first and we have

$$a^2+x^2+z^2 \cos.{}^2a+y^2 \sin.{}^2a=r^2. \qquad (2)$$

I shall also, for brevity, suppose the equations of the remaining lines, when referred to the same co-ordinates, to be

$$y'=a'x'+b',\ z'=c'x'+d',$$

as given by *Dr. Bowditch* in his solution of the same question, (referred to above) by his process, or by considering that r is the shortest distance from the point (x,y,z). To the line I have $x'=\dfrac{x+a'(y-b')+c'(z-d')}{1+a'^2+c'^2}$, put $x+a'(y-b')+c'(z-d')=p$; then $x'=\dfrac{p}{1+a'^2+c'^2}$; $\therefore$ $x-x'=x-\dfrac{p}{1+a'^2+c'^2}$, $y-a'x'-b'=y-b'-\dfrac{a'p}{1+a'^2+c'^2}$; and $z-c'x'-d'=z-d'-\dfrac{c'p}{1+a'^2+c'^2}$;

these give $r^2=(y-b')^2+(z-d')^2+x^2-\dfrac{p}{1+a'^2+c'^2}$ (3)

From (2) and (3), I derive

$$a^2=y^2 \cos.{}^2a+z^2 \sin.{}^2a-2(yb'+zd')+b'^2+d'^2-\frac{p^2}{1+a'^2+c'^2};$$

hence, $p=x+a'(y-b')+c'(z-d')=\sqrt{\{(y^2 \cos. a^2+z^2 \sin.{}^2a-2(yb'+zd')+b'^2+d^2-a^2\}\times e^2}$, assuming $(e^2=1+a'^2+c'^2)$ $\therefore$ $bx=\sqrt{\{(y^2 \cos.{}^2a+z^2 \sin.{}^2a-2(yb'+zd')+b'^2+d'^2-a^2\ \times e^2b^2\}}-\big(a'(y-b')+c'(z-d')\big)b=yz$ (4) (for (1) gives $bx=yz$). From (1) and (2) I also derive $\dfrac{a^2b^2+y^2z^2+(z^2 \cos.{}^2a+y^2 \sin.{}^2a)b^2}{b^2}=r^2$ but $r=$ min. $\therefore$ $a^2b^2+y^2z^2+(z^2 \cos.{}^2a+y^2 \sin.{}^2a)b^2=$ min. $\therefore$ $z^2ydy+y^2zdz+b^2 \cos.{}^2azdz+b^2 \sin.{}^2a\ ydy=o.$ (5) Take the differential of (4) and I have

$$\left(\frac{\cos.{}^2a\ ydy+\sin.{}^2a\ zdz-(b'dy+d'dz)}{q}\right)eb-(a'dy+dzc')b=dyz$$

$+dzy$: or $\big(\cos.{}^2a\ ydy+\sin.{}^2a\ zdz-(b'dy+d'dz)\big)eb-(a'dy+{}$ ${}+cdz)bq=dy\ zq+dz\ yq$ (6) $\big(q=\sqrt{\{y^2 \cos.{}^2a+z^2 \sin.{}^2a-2(yb'+zd')+b'^2+d'^2-a^2\}}\big)$; but (5) gives $dy=-\dfrac{z}{y}\times\dfrac{y^2+b^2 \cos.{}^2a}{z^2+b^2 \sin.{}^2a}dz.$ This substituted in (6) gives, by reduction,

$\left((\sin.{}^2 az - d')eb - c'bq\right) \times y \times (z^2 + b^2 \sin.{}^2 a) - \left((\cos.{}^2 a\, y - b')eb - a'bq\right) z \times (y^2 + b^2 \cos.{}^2 a) = qy^2 \times (z^2 + b^2 \sin.{}^2 a) - qz^2(y^2 + b^2 \cos.{}^2 a)$ (7) equations (4) and (7) contain the solution of the question. (4) cleared of radicals, gives an equation of the 4th order in terms of y and z; also, (7) cleared of radicals, gives an equation of the 10th order in terms of y and z: construct, then, the two curves, having the resulting equations for their equations, the two curves being referred to the same rectangular co-ordinates, then will the values of y and z, belonging to the points of intersection of the two curves, be the only values of y and z, which can apply to the question; and it is easy to see, that all the values thus found give minima; for $(z^2 + b^2 \sin.{}^2 a) \times (y^2 + b^2 \cos.{}^2 a) > z^2 y^2$ and $z^2 + b^2 \sin.{}^2 a$; $y^2 + b^2 \cos.$ are each positive, which are the conditions for the minima. Note. There are particular cases of this question which are very simple, viz. when the given lines intersect in one point, in which case the minimum sphere $= o$. Also, when the straight lines intersect so as to form a plane triangle, when the minimum radius is manifestly equal to the radius of the inscribed circle. Again, when the three straight lines are parallel, but not in one plane, and when the minimum radius = the radius of a circle which would circumscribe a triangle, having its angles upon the three straight lines and its plane perpendicular to them. But these cases are so obvious, that I shall not consider them further.

QUESTION XX. (150.)—*By Mr. Matthew Collins, Professor of Mathematics, Limerick, Ireland.*

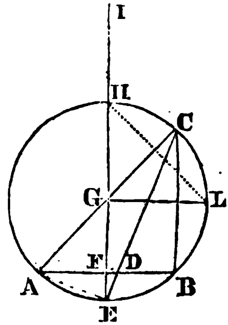
Given the hypothenuse of a right angled triangle, to construct it when the product of the nth power of the bisector of one acute angle, and the mth power of the segment of the other leg between the bisector and the other acute angle, is a maximum, m and n, bring any integers, and show previously a strictly geometrical analysis of every step of the composition.

FIRST SOLUTION—*By Mr. M. O'Shannessy, A. M.*

For simplicity, I shall consider a right angled triangle similar to the required one, whose hypothenuse is unity, of which let x and y represent the perpendicular and its adjacent segment of the base; then $\frac{y}{x}$ = the segment adjacent to the hypothenuse, and $\sqrt{(x^2+y^2)}$=the bisector; then $\sqrt{(x^2+y^2)^n} \times \left(\frac{y}{x}\right)^m$ must be a maximum, the differential of which equated to zero, having previously substituted for y its equal $x\sqrt{\left(\frac{1-x}{1+x}\right)}$ and there results $x^2 + \frac{2m+n}{n} x = 2$; a quadratic, which may be constructed as

usual. *Otherwise*. Let a = the hypothenuse, and proceeding as above, we get $x^2 - \frac{(m+n)a^3+m}{(m-n)a^2-m}x - \frac{2a^4n}{(m-n)a^2-m} = o$; a quadrature, which may also be constructed.

SECOND SOLUTION.—*By Mr. Benjamin Hallowell.*



Analysis. Let ABC represent the required triangle, CD the line bisecting the angle ACB, and GF a line parallel to CB, from the centre of the circumscribing circle; then GF and CD being produced will meet at the circumference in E. Now $AF^2 = AG^2 - GF^2$; $AE^2 = EH.EF = 2AG(AG-GF)$; $EC^2 = AC^2 - AE^2 = 2AG(AG+GF)$. By similar triangles $EC^2 : AC^2 :: CB^2(4GF^2) : CD^2 = \frac{8AG.GF^2}{AG+GF}$ and $EC^2 : AC^2 :: AE^2 : AD^2 = \frac{4AG^2(AG-GF)}{AG+GF}$. Hence as $CD^n . AD^m$ is to be a maximum, we have (rejecting constants) $\frac{GF^n(AG-GF)^{\frac{m}{2}}}{(AG+GF)^{\frac{m+n}{2}}}$ to be a maximum which by the known principles of max. et min. will be when $2AG^2 = (GF + \frac{n+2m}{n} AG)\ GF$.

Composition. Draw GL perpendicular to EH, and join HL. Take GI a fourth proportional to n, $n+2m$ and AG, and produce IG to F, so that $IF.FG = HL^2$. Draw AFB parallel to GL, and draw the diameter AC. Also, join CB and CE and the thing is done. For by construction, $n : n+2m :: AG : GI = \frac{n+2m}{n}AG$. Now $2AG^2 = HL^2 =$ (by construction) $IF.FG = (GF+IG).GF = (GF + \frac{n+2m}{n}AG).GF$.

THIRD SOLUTION.—*By Henry Dose, Esq.*

The right angled triangle ABD has the angle A bisected by the line AC, and it is required to make $AC^n \times CD^m$ a maximum, m, n, and $AD = h$ being given. Put $BAD = 2z$; $\therefore BAC = CAD = z$; $AB = h.\cos.2z$; $AC = \frac{h.\cos.2z}{\cos.z}$; $CD = h.\tan.$; so that

the quantity to be a maximum is $\left(\frac{\cos. 2z}{\cos. z}\right)^n . (\tan. {}^{\prime}z)_m$. The differential of its logarithm put $= o$, gives $-2n \tan. 2z + n \tan. z + \frac{m}{\cos. z. \sin. z} = o$: Substituting $\tan. z = \frac{\sin. z}{\cos. z}$, $\tan. 2z = \frac{2 \cos. z \sin. z}{1-2 \sin. {}^2 z}$, and reducing, we shall get

$$\sin. {}^2 z = \frac{3n+2m-\sqrt{(9n^2+4mn+4n^2}}{4n}.$$

Thus, if $m = 5$, $n = 4$, $\sin. z = \frac{1}{2}$, and $z = 30^{\circ}$.

QUESTION XXI. (151.)—*By Professor Strong.*

Two trees, of given heights, stand on a side hill, which is an inclined plane, having a given inclination to the horizon; a person, whose eye is situated in the given plane at a certain point, measures the angles subtended by the trees. From these data, it is required to find the place of the observer's eye, supposing the line joining the foot of the trees is also given in length and position.

FIRST SOLUTION.—*By Mr. Charles Farquhar.*

In the solution of this elegant problem, I propose to find a curve which must be described on an inclined plane, such that the angle subtended by an object perpendicular to the horizon, at any point of the curve, shall be invariable: then the intersection of two such curves, about the two trees as poles, will determine the required point.

Let $a =$ height of one tree, $m =$ the given angle it subtends, $p =$ the given inclination of the plane, $z =$ the distance from the bottom of the tree to a point in the curve, and $\varphi =$ the angle contained between z and a line directly up the plane.

Now, by spherics, the cosine of the angle contained between the tree and the radius vector $z = \sin. p . \cos. \varphi$, and the sine of the same angle $= \sqrt{(1 - \sin. {}^2 p . \cos. {}^2 \varphi)} = x$ (for brevity); also (by Trig.) the sine of the angle contained between the tree and a line drawn from its top to the extremity of $z = x . \cos. m + \sin. m\sqrt{.(1 - x^2)}$; therefore, the radius vector $= a . \frac{x . \cos. m + \sin. m\sqrt{(1-x^2)}}{\text{sine } m} = ax . \cot. m + a\sqrt{(1-x^2)} =$ (by replacing the value of x,) $a . \cot. m\sqrt{(-\sin. {}^2 p . \cos. {}^2 \varphi)} + a \cos. \varphi . \sin. p$; again, the dist. of the tree from the centre of the curve is given $= a . \sin. p =$ (for brevity) d; also, the dist. from the tree to the curve, in a direction at right angles to the line up the plane, is given, $= a . \cot. m = l$; therefore, I have

$$z = l \sqrt{(1 - \sin. {}^2 p \cos. {}^2 \varphi)} + d \cos. \varphi;$$

which is the polar equation of the curve.

Cor. If $p=o$, then $d=o$, and $z=l$, or the curve becomes a circle.

Adopting a similar notation, the equation of the curve, about the other tree, is $z'=l'\sqrt{(1-\sin.^2 p\,.\cos.^2\varphi')}+d'\cos.\varphi'$. To determine the intersection of these two curves, let $c=$ the given distance between the trees, A and $A'=$ the given angles between c, and two parallel lines passing through the extremities of c, up the plane, then (by Trig.) I have two equations,

$$c\,.\frac{\sin.(A-\varphi)}{\sin.(\varphi+\varphi')}=z,\text{ and } c\,.\frac{\sin.(A'-\varphi')}{\sin.(\varphi+\varphi')}=z';$$

from which, and the above equations of the curves, the required point or points are determined.

SECOND SOLUTION.—*By Dr. Bowditch.*

It is evident, that if a vertical circular arch be described upon the first tree as a chord, so that the angle contained by lines drawn from the circumference to the top and bottom of the tree shall be equal to the observed angle subtended by the tree, the eye must be placed upon the surface formed by the revolution of this arch about its chord. A similar second surface being described for the second tree, the intersection of these two surfaces will be a *curve* on which the eye is placed. The intersection of this curve with the plane of the hill will be the required place of the observer.

Suppose now the rectangular co-ordinates of any point of the *first* surface to be x, y, z; x, y being horizontal and z vertical, the co-ordinates of the middle of the first tree a, b, c, the radius of the first mentioned arch r and the distance of its centre from the centre of the first tree e. The similar given quantities for the second tree being accented. Then we evidently have for the equation of the first surface

$$(x-a)^2+(y-b)^2=\{e+\sqrt{r^2-(z-c)^2}\}^2$$

and for the 2nd surface $(x-a')^2+(y-b')^2=\{e'+\sqrt{r'^2-(z-c')^2}\}^2$ the equation of the plane of the hill being $z=fy$, supposing the axis of x to be on the line of intersection of the hill with the horizontal plane.

This value of z being substituted in the two first equations they will contain only x and y. Their difference will give x in terms of z, which being substituted in either of these last equations, will give a final equation in z, from which z may be determined, and from thence we easily get x and y.

If e, e' should be o, or the angles subtended by the trees 90°, the surfaces become spherical and the two equations of the surfaces become

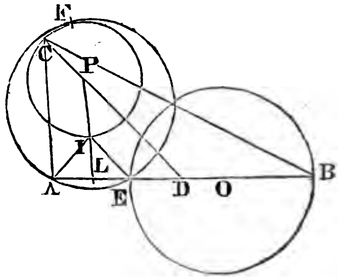
$$(x-a)^2+(y-b)+(z-c)^2=r^2$$
$$(x-a')^2+(y-b)+(z-c)^2=r'.$$

The difference of these two equations is of the first degree in x, and z. This and the equation of the plane give x, y in terms of the first degree in z. Substituting these in either equation of the surface gives z by a quadratic equation.

QUESTION XXII. (152.)—*By Mr. J. H. Swale, Liverpool, Eng.*

To constitute a triangle, having one extremity of its base at a point given in position, and the other angular points posited on the peripheries of two circles, given in position and magnitude: when the base and the vertical angle, or the base and the sum of the squares of the sides, are either given, or a maximum or minimum.

FIRST SOLUTION.—*By Mr. Charles Farquhar.*



Let A be the given point, O, and P the centres of the given circles. From A, through O, draw AB, then by Prob. XLV. Simp. Geom. describe a circle through A, and E, so as to touch the circle P, in F; join AF, EF, and AEF will be a triangle, whose base and vertical angle are a minimum. Also, through A and B describe a circle to touch the circle P, on the lower part, and we have the maximum base and vertical angle. In the case when the base and verticle angle are given, from A, in the circle O, apply the given base, on which describe a segment to contain the vertical angle, and the point of intersection with the circle P, will be the vertex of the triangle; all of which is too evident to need a demonstration.

When the base and sum of the squares of the sides are given, see Simpson's Exercises for a construction.

When these are to be a maximum or minimum, bisect AB, and AE, in D, and L; join DP, and LP, cutting the circle in C, and I, the vertices of the required triangles. For, $AC^2+CB^2=2.AD^2+2.DC^2$; therefore, because AD is constant, DC is a maximum, as per construction*; and in a similar manner, LI is a minimum, as also AE.

SECOND SOLUTION.—*By Mr. Benjamin Hallowell.*

The base is evidently known in all the cases of this problem; for when it is a given quantity, apply it from the given point to the periphery of either of the circles. When it is to be a maximum or minimum through the given point, draw a diameter to each of the circles, then the greatest distance between the given point and either extremity of either diameter, will be the maximum base, and the least distance, the minimum base.

* $AC^2+CB^2 \propto DC^2$, therefore, as the latter is a maximum, the former is.

A solution to the question when the vertical angle is a given quantity, or a maximum or minimum, is given by Dr. Hutton in answer to the prize question in the Ladies' Diary for 1737. Simpson, in his Geometry, Prob. XII. of the Construction of Geometrical Problems, solves the question when the sum of the squares of the sides is *given ;* and John Gummere, in question XII. of the Analyst, proposes and solves the question when the sum of the squares of the sides is a maximum or minimum; which are all the cases of this compound problem.

Question XXIII. (153.) OR PRIZE QUESTION.

By Dr. Anderson.

To determine the motion of a uniform heavy inflexible circular plate, placed originally in a position nearly vertical upon a horizontal plane, and then impelled in the direction of its plane ; supposing the friction just sufficient to make the plate's circumference tend to roll without sliding, along its variable projection on the horizontal plane.

Prize Solution.—*By Mr. Eugenius Nulty, Philadelphia.*

Let the circular plate be referred to rectangular axes passing through its centre of gravity, and at any instant during its motion, let α, β, γ be the velocities of any element, dm relatively to this centre and in the directions of the axes; α', β', γ' the corresponding velocities of the particle in contact with the horizontal plane; c, the vertical distance of the same particle from the origin of the axes, and g the force of gravity.

The velocity of the centre of gravity of any body rolling on a surface is evidently equal, and in a direction contrary to the velocity of the changing point of contact. The absolute velocities of the element dm in the directions of the axes are, therefore, $\alpha-\alpha'$, $\beta-\beta'$, $\gamma-\gamma'$; by virtue of which and known principles, there immediately results for half the living force of the rolling plate, diminished by the sum of the actions of gravity,

$$F=\tfrac{1}{2}\mathrm{S}dm\,.\,(\alpha^2+\beta^2+\gamma^2)+\tfrac{1}{2}m\,.\,(\alpha'^2+\beta'^2+\gamma'^2)-mgc,$$

an expression which will enable us to determine the motion of any body whatever, subjected to gravity, and compelled to roll in contact with a surface in consequence of friction and resistance.

Conceive the axes to which we have referred the plates, to coincide with its principal axes, and let the co-ordinates of dm in the plane of the plate be a, b ; the co-ordinates of the point of contact, a', b' ; the angular velocities round the principal axes, p, q, r ; the angles of which these velocities are known functions, φ, ψ, ω ; and the radius of the plate ρ. We shall then have in case of a material plane surface $\alpha=-br$, $\beta=ar$, $\gamma=bp-aq$, and since the values of α', β', γ' are similar, we shall have by substitution

$F=\frac{1}{2}Sdm\{(a^2+b^2)r^2+a^2q^2+b^2p^2\}+\frac{m}{2}\{(a'^2+b'^2)r^2+(b'p-a'q)^2\}$ $-mgc$. This expression in its present form will apply to a disk of any figure whatever rolling on a surface; but when this figure is a circle, and when the surface on which it rolls is a horizontal plane, we have evidently $Sa^2dm=Sb^2dm=\frac{1}{4}\rho^2m=A$; $a'=\rho\sin.\varphi$, $b'=\rho\cos.\varphi$, and $c_{,}=\rho\sin.\omega$, by virtue of which and the known values of the angular velocities p, q, r we have

$$F=\tfrac{1}{2}\{C(\varphi'+\cos.\omega.\psi')^2+A\sin.\omega^2.\psi'^2+B\omega'^2\}-C'\sin.\omega\ ; \quad (1)$$

in which $\varphi'=\frac{d\varphi}{dt}$, $B=A+\rho^2m$, $C=2A+\rho^2m$ and $C'=\rho mg$.

If now the variations of this expression be taken according to the formula

$$\frac{d}{dt}.\left\{\frac{\delta F}{\delta\varphi'}\delta\varphi+\frac{\delta F}{\varphi\psi'}\delta\psi+\frac{\delta F}{\delta\omega'}\delta\omega\right\}-\frac{\delta F}{\delta\omega}\delta\omega\ ;$$

ann if we observe, that conformably to the present requisitions of the question, the variations $\delta\varphi$, $\delta\psi$, $\delta\omega$ are independent, we shall obtain

$$d.(\varphi'+\cos.\omega.\psi')\text{ or }dr=0\ ; \quad (2)$$
$$d.(A\sin.\omega^2.\psi'+Cr\cos.\omega)=0\ ; \quad (3)$$
$$B\frac{d\omega'}{dt}+Cr\sin.\omega.\psi'-A\sin.\omega\cos.\omega.\psi'^2+C'\cos.\omega=0. \quad (4)$$

These are the equations by means of which the motion of the circle may be completely determined when the resistances first proposed are excluded; but were these forces to be taken into consideration, we should have merely to increase equation (2) by a term of the form srm and equation (4) by $s'\omega'^2$.

Let us proceed to integration. The first of these three equations immediately gives

$$\varphi+\cos.\omega.\psi',\text{ or }r=h, \quad (5)$$

which shows that the velocity of the plate round the axis perpendicular to its plane, is constant. The second equation also, gives

$$A\sin.\omega^2.\psi'+C\cos.\omega=C\,h\cos.i, \quad (6)$$

in which i represents the primitive value of the angle ω, or of the inclination of the plate to the horizontal plane; and from which we infer, that the sum of the elements of the plate respectively multiplied by the areas traced by their projections on the horizontal plane, is constant, and equal to the moment of impulsion $Ch\cos.i$. Lastly, the equation (4) becomes, by virtue of (5) and (6),

$$B\frac{d\omega'}{dt}+\frac{C^2h^2(\cos.i-\cos.\omega)}{A\sin.\omega}-\frac{C^2h^2(\cos i-\cos.\omega)^2}{A\sin.\omega^3}+C'\cos.\omega=0\cdot(7)$$

This equation multiplied by $d\omega$, admits of an integral which is easily seen to be equivalent to (1), and by means of which $d\psi$ and $d\varphi$ may be expressed in terms of ω; but the complete integration

of these expressions in finite terms corresponding to all values of the inclination ω will be found impossible in the present state of analysis, and can be effected only by having recourse to series and the usual methods of approximation.

Limiting our views, therefore, to the case in which ω is nearly equal to $\frac{\pi}{2}$ and the plate nearly vertical, let us assume the primitive inclination $i = \frac{\pi}{2} - i_{,}$ and at the end of the term t, let $\omega = \frac{\pi}{2} - (i_{,} - \theta)$. The angle $i_{,}$ and θ being then considered as extremely small, we shall have sin. $\omega = 1$, cos. $\omega = i_{,} - \theta$, &c. and by substitution, equation (7) becomes

$$\frac{d^2\theta}{dt^2} + \frac{C^2h^2 - AC'}{AB}\theta + \frac{C'i_{,}}{B} = 0,$$

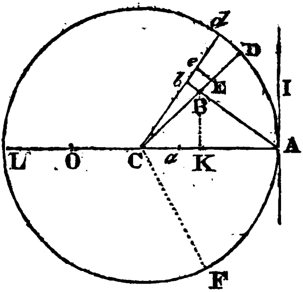
the corrected integral of which by assuming $ABk^2 = C^2h^2 - AC'$ and $BKk^2 = C'i_{,}$, is exactly found to be $\theta = K(1 - \cos. kt)$.

From this expression, it may be inferred, that when $C^2h^2 < 2AC'$, the angle θ cannot remain small, and the plate will necessarily descend to the horizontal plane; and that when these quantities are equal, or when the first exceeds the second, oscillations will take place, and the excursions of the plate will be confined to one side of the vertical passing through the changing point of contact when $C^2h^2 > 3AC'$, but will extend to this vertical when these quantities are equal, pass it when the first is less than the second, and will become the greatest possible when $C^2h^2 = 2AC_{,}$. If we therefore assume $C^2h^2 = (2+n)AC'$, and suppose n any positive number whatever, we shall have $k^2 = (1+n) \cdot \frac{C'}{B}$, $\omega = \frac{\pi}{2} - i_{,} + K(1 - \cos. kt)$, and the time of oscillation $t = \pi k$. These expressions may be said to answer the problem according to the preceding suppositions, since by means of them we immediately find $\psi = \frac{ChK}{Ak}(kt - \sin. kt)$ from equation (6), and since $\varphi = ht$ by virtue of (5), and $k^2(CK\varphi - A\psi)^2 + C^2h^2(\theta^2 - 2k\theta) = 0$ by eliminating t, every circumstance relating to the oscillations of the plate must be considered as fully determined.

Second Solution to the Prize Question.

By Nathaniel Bowditch, LL.D.

The curve which is described on the horizontal plane by the circumference of the plate in its motions may be considered as a groove, in which the plate is compelled to roll without sliding; the re-action of the groove taking the place of the friction to prevent sliding.



Suppose now the radius of the plate CA=R, and for greater generality we shall suppose it to be in the form of an annulus, whose inner radius Ca=r. At the commencement of the motion let F be the point in contact with the plane, and at the end of the time t, let A be the point in contact, making the arch FA $=\text{R}\times s$, and let the inclination of the plate to the horizontal plane be $90°-u$, u being by hypothesis a very small angle.

Draw the two radii CBD, Cbd, so that the angle FCD$=\sigma$, DCd=$d\sigma$; about the centre C, with the radii CB$=\varrho$, CE$=\varrho+d\rho$, describe the arcs Bb, Ee, then the particle of the plate BEeb which we shall call dm is $=\varrho d\varrho \,.\, d\sigma$. Draw BK perpendicular to CA, and let BA$=\varrho'$, AK$=\varrho''=\text{R}-\varrho \,.\, \cos.(\sigma-s)$.

Then the velocity of the centre of gravity of the plate is $\frac{\text{R}ds}{dt}$ being the same as that with which the successive points of the circular plate come in contact with the groove. And as the point A is, for an infinitely small moment of time at rest, while all the other points of the plate revolve about it as a centre, the velocity of the point B will be equal to $\varrho' \,.\, \frac{ds}{dt}$, in a direction perpendicular to AB. The differential of this divided by dt is to be multiplied by dm and by the virtual lineola $\varrho'\delta s$ to obtain the corresponding term of the fundamental equation of p. 195 Ed. of La Grange's *Mec. Anal.* $\frac{d.(\varrho'ds)}{dt^2}.\, \varrho'dm \,.\, \delta s$, or as it may be written $\frac{d.(\varrho'ds)^2}{2dt^2ds}\,.\, dm.\delta s$.

Again, the angular velocity of the plane of the circle about the tangent AI, on account of the variation of u is $\frac{du}{dt}$, this multiplied by the distance ϱ'' of the point B or K from that tangent gives its velocity $\frac{\varrho''du}{dt}$; its differential divided by dt and multiplied by the particle dm and the virtual lineola $\varrho''\delta u$ may be put under the form $\frac{d\,.\,(\varrho''du)^2}{2dt^2du}.\, dm \,.\, \delta u$.

It is not necessary to notice the motion of the plate about the axis LA, for this motion being of the order uds, and when multiplied by the virtual lineola $u\delta s$, it becomes of the order u^2 and may be neglected.

Finally, if through the point B, we suppose a vertical line z to be drawn to the horizontal plane xy, the effect of the gravity g in La Grange's Formula, will be $-g\delta z\,.\,dm$, but $z = \varrho''\cos.\,u$, whence $\delta z = \delta\varrho''\cos.\,u - \varrho''\delta u\sin.\,u$, and, by retaining only the first power of u, $\delta z = \delta\varrho'' - \varrho'' u\,.\,\delta u$, and since $\delta\varrho'' = -\varrho\delta s\,.\,sin\,(\sigma - s)$, we get for the term $-g\delta z dm$ the expression $\big(g\varrho\sin.(\sigma - s)\,.\,dm \times \delta s + g\varrho'' u\, dm\,\delta u.\big)$

The sum of these three terms being found and the integral taken for all the particles of the plate, we shall get by putting the coefficients of δs and δu separately equal to 0, the two following equations, using $dm = \varrho d\varrho\, d\sigma$,

$$\int\left(\frac{d\,.\,(\varrho'^2 ds^2)}{2dt^2.ds} + g\varrho\sin.\,(\sigma - s)\right).\,\varrho d\varrho d\sigma = 0.$$

$$\int\left(\frac{d\,.\,(\varrho''^2 du^2)}{2dt^2 du} + g\,\varrho'' u\right)\varrho\,.\,d\varrho\,.\,d\sigma = 0.$$

The integrals relative to σ are to be taken from $\sigma = 0$ to $\sigma = 2\pi$, $\int d\sigma = 2\pi$, and the times depending on cos. $(\sigma - s)$ and its multiples varied, so that instead of ϱ'^2 where general value is $R^2 - 2R\varrho\,.\cos.\,(\sigma - s) + \varrho^2$, we may write $\varrho'^2 = R^2 + \varrho^2$, and for ϱ''^2 we may put $R^2 + \frac{1}{2}\varrho^2$, neglecting the terms depending on these angles. Taking then the integrations relative to ϱ from $\varrho = r$ to $\varrho = R$, we get $\int\varrho'^2\varrho d\varrho = \frac{1}{2}R^2(R^2 - r^2) + \frac{1}{4}(R^4 - r^4)$; $\int\varrho''^2\varrho d\varrho = \frac{1}{2}R^2.(R^2 - r^2) + \frac{1}{8}(R^4 - r^4) = (R^2 - r^2)\,(\frac{5}{8}R^2 + \frac{1}{8}r^2)$, and $\int\varrho d\varrho = \frac{1}{2}(R^2 + \varrho^2)$. Substituting these and reducing the preceding equations become

$$\frac{dds}{dt^2} = 0.$$

$$\frac{ddu}{dt^2} + \frac{gR\,u}{\frac{5}{4}R^2 + \frac{1}{4}r^2}. = 0.$$

This last may be put under a more simple form by observing that if O be the centre of oscillation of the plate or annulus, about the tangent AI, perpendicular to the plane of the figure, and AO $= l$, we shall have $l = \frac{5}{4}R + \frac{r^2}{4R}$ which being substituted in the last equation it becomes $\frac{ddu}{dt^2} + \frac{g}{l}.\,u = 0$, whose general integral is $u = b\,.\cos.\,(t\sqrt{\frac{g}{l}} + c)$ which gives $\frac{du}{dt} = -b\sqrt{\frac{g}{l}}.\sin.(t\sqrt{\frac{g}{l}} + c)$ and this being per question 0 when $t = 0$ (because the initial velocity in the direction of the plane is 0) we get $c = 0$. Supposing moreover that s and t begin together, the first of the

above equations gives $s = \frac{at}{R}$ and we finally have,

$$R \,.\, s = at \tag{1}$$

$$u = b.\ \cos.\ (t\sqrt{\tfrac{g}{l}}) \tag{2}$$

Hence it appears that while the plate moves on the plane with an uniform velocity $R\frac{ds}{dt} = a$, the inclination u oscillates about the vertical from b to $-b$, b being the inclination of the plane to the vertical at the commencement of the motion.

Suppose T to be the time of one vibration of the plate from the greatest positive inclination b to the greatest negative value $-b$. Thus by equation (2) will give $T\sqrt{\frac{g}{l}} = \pi$, whence

$$T = \pi \sqrt{\frac{l}{g}} \tag{3}$$

Now by page 31, vol. i. *Mec. Cel.* this exactly agrees with the time of vibration of a simple pendulum whose length is l. Therefore the inclination u varies from $+b$ to $-b$ in the same time as the plate itself would freely vibrate by the force of gravity when suspended by the tangent AI, and made to revolve perpendicular to its plane.

To find the horizontal groove or curve line described by the point of contact of the plate with the horizontal plane, we may observe that dt being supposed constant the equation (1) will give Rds constant. Now if we take two consecutive equal elements Rds of the circumference of the plate (considered as a polygon of an infinite number of sides) and continue the first side on the horizontal plane by an equal quantity Rds, the distance of this point from the end of the second element will be $\frac{\overline{Rds}|^2}{R}$ or simply Rds^2, and this multiplied by sin. u or u, will give the horizontal projection of this distance equal to $Ruds^2$, and this quantity may be put equal to ddy if we take the axis of x parallel to the initial direction of the plate, and the axis of y perpendicular be that of x, rejecting terms of the order u^2. Substituting in this value of ddy the above value of u and ds^2 we get $\frac{ddy}{dt^2} = \frac{a^2}{R} \,.\, b \,.\, \cos.\ t\sqrt{\frac{g}{l}}$ whose integral taken to vanish when $t\sqrt{\frac{g}{l}} = 90^\circ$, is

$$y = -\frac{a^2bl}{gR} \,.\, \cos.\ t\sqrt{\frac{g}{l}} \tag{4}$$

If we suppose the velocity a to be such as would be acquired by falling freely by gravity through the space h, we shall have $2gh = aa$, substituting this and putting for t its value $\frac{Rs}{a}$ or what is

very nearly equal to it $\frac{x}{a}$ differing only by terms of the order u^2, on account of the smallness of u, y, b, it will become $y = -\frac{2hlb}{R} . \cos. \frac{x}{\sqrt{2hl}}$. which is the required equation of the curve, and if we alter the origin of x, y by writing $x + \frac{1}{2}\pi . \sqrt{2hl}$ for x, it will become

$$y = \frac{2hlb}{R} . \sin. \frac{x}{\sqrt{2hl}}$$

So that the curve described by the plate on the horizontal plane is a curve of the nature of the *curve of sines.*

Cor. 1. Hence it appears that the values of s, u, y have the same form for a complete circular *plate* as for an *annulus* or hoop, the only difference consists in the value of l. In the former case of a whole plate let this be l', in the latter if a thin hoop let it be l''. The values of l', l'' are found from $l = \frac{5}{4}R + \frac{1}{4} . \frac{r^2}{R}$ by putting in the first case $r = 0$, in the last $r = R$, whence we get

$$l' = \frac{5}{4} R$$
$$l'' = \frac{6}{4} R$$

and putting T′, T″ for the corresponding values of T, we shall have by equation (3)

$$T' = \frac{1}{2}\pi \sqrt{5 . \frac{R}{g}}$$

$$T'' = \frac{1}{2}\pi \sqrt{6 . \frac{R}{g}}$$

whence $\quad T' : T'' :: \sqrt{5} : \sqrt{6}$

So that a hoop is about one tenth part longer in performing its vibrations than a plate.

Third Solution.—*By Dr. Anderson.*

Let a denote the radius of the plate, μ its mass, A, B, A', B', its principal moments of inertia referred to the point of contact and to the centre of gravity; x, y, z, x', y', z', the space co-ordinates of these points; θ, ϕ, the small angles which the normal axis of the plate makes with the planes xy and yz; l, m, n, the velocities of rotation round the axes x, y, z; L, M, N, the angles described with these velocities, and so reckoned that M increases with x. Finally, let g denote gravity, and t the time.

The conditions of the question are comprised in the three equations,

(1) $\qquad dx - a dM = 0$

(2) $\qquad dy + \phi dx = 0$

(3) $\qquad a d\phi + \theta dx = 0$

We have, also, the relations,

$$(4) \quad dx' = dx, \qquad dy' = dy + ad\theta, \qquad dz' = a\theta d\theta$$
$$(5) \qquad d\varphi = d\mathcal{N} - \theta dM$$
$$(6) \qquad d\theta = \varphi dM - dL$$

Equations (1), (3), (5), give $d\mathcal{N} = o$. The general formula

$$\frac{d^2x'}{dt^2}\delta x' + \frac{d^2y'}{dt^2}\delta y' - g\delta z' + P\delta L + Q\delta M + R\delta \mathcal{N} = o,$$

by means of these equations, becomes

$$o = \left\{ \mu a \frac{d^2y'}{dt^2} - \mu a g\theta - \frac{d\,\{A'l + (B' - A')\,m\varphi\}}{dt} \right\} \delta L$$
$$+ \left\{ \mu a \frac{d^2x'}{dt^2} + \frac{d\,.\,Bm}{dt} \right\} \delta M$$

Whence $m =$ constant, and

$$\frac{d^2\theta}{dt^2} + \frac{Bm^2 - \mu a g}{A}\theta = o.$$

It appears, therefore, that the oscillatory motion of the plate will be defined by

$$\theta = \theta' \,.\, \cos.\, et$$
$$\varphi = \varphi' \,.\, \sin.\, et$$

where $e\varphi' + m\theta' = o$, e being the square-root of the above coefficient of θ. The normal axis of the plate (abstracting from the motion of the centre of gravity) will, therefore, keep constantly describing an elliptical cone, whose angles are $2\,\theta'$ and $2\,\varphi'$. The motion of the point of contact and centre of gravity will be defined by the equations,

$$x = amt \qquad\qquad x' = amt$$
$$y = -a\theta' \frac{m^2}{e^2} \cos.\, et \qquad y' = -a\theta' \frac{m^2 - e^2}{e^2} \cos.\, et$$

and their paths will be harmonic curves.

If the condition of rolling along the projection be omitted, and the plate, or more generally a spheroid of revolution be simply considered as subject to roll and spin without sliding, all the phenomena of motion may be deduced from the general results I have given in a memoir on the motions of solids on surfaces, read (Jan. 4, 1828,) before the Philosophical Society of Philadelphia, and shortly to be published in their Transactions. The conditions of perfect rolling are contained in the three equations,

$$dx' = (z' - z)dM - (y' - y)d\mathcal{N}$$
$$dy' = (x' - x)d\mathcal{N} - (z' - z)dL$$
$$dz' = (y' - y)dL - (x' - x)dM.$$

In this case the value of e^2 will be found to be

$$\frac{BB'm^2 - \mu A'\lambda g}{AA'};$$

λ being the difference between the greatest and least radii of curvature at the balancing points of the spheroid. The paths of the

point of contact and centre of gravity are still harmonic curves, but not the same as before. The plate does not roll along its variable projection, but along a curve whose curvature at the point of contact is in a constant ratio to the curvature of the plate's projection at the time. In the present case, there will be oscillations also round the normal; in the former case, there was none.

For a uniform plate the above values of e^2 will be

$$\frac{6}{5}m^2 - \frac{4\,g}{5\,a} \qquad \text{and} \qquad \frac{12}{5}m^2 - \frac{4\,g}{5\,a}$$

according as the hypothesis of rolling along the variable projection be included or not.

ACKNOWLEDGMENTS, &c.

The following gentlemen favoured the Editor with solutions to the questions in Article XVIII. No. 8. The figures to the names refer to the questions answered by each as numbered in that article.

Dr. Bowditch, Boston; Mr. Eugenius Nulty, Philadelphia; Professor Strong, Rutgers College, New-Brunswick, N.J.; Dr. Adrain, Philadelphia; Dr. Henry J. Anderson, Columbia College, N.Y.; Henry Dose, Esq. Natches, Miss.; Messrs. Charles Farquhar, Alexandria, D. C.; Benjamin Hallowel, Alexandria, D. C.; and Michael O'Shannessy, A.M., N.Y., each answered all the questions.

Messrs. Ingersoll J. Bowditch, Boston; James Macully, N. Y.; Omicron, N. C.; James Divver, South Carolina College; James Pierce, Cambridge College, Mass.; and William H. Sidell, N. Y. each answered all but the Prize Question.

Mr. John D. Williams, Dighton, Mass. answered all but 22 and 23; Mr. Marcus Catlin, all but 19, 21, and 23; Mr. Nathan Brown, all but 18, 19, 21, and 23; Mr. Solomon Wright, Bucks County, Penn. all but 15, 17, 21, 22, 23; Mr. William J. Lewis, Philadelphia, all but 15, 19, 20, 21, 23; Mr. John F. James, Philadelphia, answered the first seventeen questions; Mr. Gerardus B. Docharty, all but 15, 18, 19, 20, 21, 23; Mr. John Swinburne, Brooklyn, the first seventeen; Mr. A. Hammond, Brooklyn, all but 13, 18, 19, 20, 21, 22, 23; Mr. Edward Devoy, New-York, all but 15, 17, 18, 19, 20, 21, 22, 23; Mr. Thomas I. Megear, Wilmington, all but 10, 15, 17, 19, 20, 21, 23; Mr. Enoch Lanning, all but 15, 16, 18, 19, 20, 21, 22, 23; Hypathia, Penn. answered, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 18; Mr. Silas Warner, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16; Mr. John M. Wilt, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16; Mr. Henry Doyle, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16; Mr. James Docharty, Flushing, L.I. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16; Mr. George W. Taylor, Westchester, N.Y. 1, 2, 3, 4, 5, 6, 7, 9, 12, 14, 16; Mr. James Sloane, Middletown Academy, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 14; Mr. Michael Floy, jun. N.Y. 1, 2, 3, 4,

Mr. Henry R. Lott, N.Y. 1, 2, 3, 4, 5, 7, 8, 9, 14; Mr. S. Laws, Wilmington, 1, 2, 3, 5, 6, 9, 12, 16; Mr. George Evans, N.Y. 1, 2, 3, 4, 5, 7, 9, 10; Mr. J. H. Jenkins, 2, 3, 5, 7, 9, 14, 16; Mr. William Lenhart, 8, 14, 16, 17, 20; Mr. John C. Reid, N. Y. 2, 3, 5, 9, 16: Mr. William Vodges, 1, 2, 3, 5, 6, 9, 10; Mr. B. Hussey, 2, 3, 5, 9, 21; Mr. John B. Newman, jun. 1, 2, 3; and Mr. J. S. Vandegraff, Lexington, Ken. 18.

The prize for the best solution of the prize question No. 23 is awarded to Mr. Eugenius Nulty, Philadelphia.

The solutions of Professor Strong, Rutgers College; and Robert Adrain, LL.D., Professor of Mathematics, &c. in the University of Pennsylvania, were perfectly general, but the Editor did not receive them till after the prize was awarded.

ARTICLE XIX.

NEW QUESTIONS

TO BE RESOLVED BY CORRESPONDENTS IN No. X.

QUESTION I. (154)—*By Mr. S. Hammond, Brooklyn.*

Given $\left\{\begin{array}{l} x^4-y^4=1280 \\ x^3y+xy^3=480 \end{array}\right\}$ to find the values of x and y.

QUESTION II. (155)—*By Mr. John Swinburne.*

Given $\left\{\begin{array}{l} y^6-x^4=230\,40000 \\ x^2y^3-2000x^2=4800y^3 \end{array}\right\}$ to determine the values of x and y by a quadratic.

QUESTION III. (156)—*By the same.*

Given $\left\{\begin{array}{l} x^6y^3-x^{12}=9728 \\ x^2y^5+x^8y^2=40320 \end{array}\right\}$ to determine the values of x and y.

QUESTION IV. (157)—*By Mr. Michael Floy, jun. N.Y.*

It is required to find two numbers, such that their difference and the difference of their cubes shall be rational squares.

QUESTION V. (158)—*By Mr. Solomon Wright.*

Required the height of a solid of a pyramidal form, the three sides of the base being 13, 14, and 15 respectively; and the three angles of the sides taken at the vertex are found to be 30°, 40°, and 50°.

QUESTION VI. (159)—*By Mr. Farrand N. Benedict.*

Given $x^n(x^n+y^n+z^n+v^n)=a$, $y^n(x^n+y^n+z^n+v^n)=b$, $z^n(x^n+y^n+z^n+v_n)=c$, and $v^n(x+y^n+z^n+v^n)=d$, to determine the values of x, y, z, v.

QUESTION VII. (160)—*By the same.*

Make $16x^5+8x^4+8x^3+5x^2+2x+1$, a rational square.

QUESTION VIII. (161)—*By Mr. Jacob Borton.*

A circle and its diameter being given in position and magnitude, it is required to draw a tangent to the circle at one end of the diameter and a line from the other end to meet the end of the tangent; such that the part of the line intercepted by the circle shall be equal to the tangent.

QUESTION IX. (162)—*By Mr. James Sloane, N.J.*

Given two right lines in position forming an acute angle and a point adjacent to them; to describe a circle that shall have its centre in one of the lines, it shall pass through the given point, and to which the other line shall be a tangent.

QUESTION X. (163)—*By Mr. Thomas I. Megear.*

It is required to find an arc, such that the rectangle of its versed sine and the versed sine of its complement shall be a maximum.

QUESTION XI. (164)—*By Mr. James Diver.*

Given the base, the difference of the angles at the base, and the line that bisects the vertical angle; to determine the triangle.

QUESTION XII. (165)—*By Mr. William H. Sidell.*

The base, difference of the squares of the sides, and the sum of the tangents of the angles at the base being given; to construct the triangle.

QUESTION XIII. (166)—*By Mr. James Macully.*

Given the sum of the two sides containing the vertical angle of a plane triangle, and the radius of its inscribed circle to construct it; when that part of the line bisecting the vertical angle included between the base and the centre of the inscribed circle is a maximum.

QUESTION XIV. (167)—*By Ομικρον, of N. C.*

A gentleman of North Carolina having purchased a quantity of land in the form of a triangle, and being unable to select a spot suitable for building, has directed a surveyor to lay out a triangular lot for this purpose by running lines parallel to the several sides of the triangle in such a manner, that the whole together with the two parts into which it is divided by the parallel lines, may succesively constitute an arithmetical, a geometrical, and an harmonical ratio. Required the ratio of the dividing lines to their opposite sides, and also the dimensions of the triangular lot, those of the whole being known.

QUESTION XV. (168)—*By Henry Dose, Esq.*

It is required to prove that, if x be a prime number, $v^n = x^n \pm y^n$ is or is not rationally impossible, which has never been proved satisfactorily.

QUESTION XVI. (169)—*By Mr. Charles Farquhar.*

If on the ordinate PM produced, of any curve AM, the distance MM′ be taken always equal to the abscissa AP; if AM, and AM′ be joined, then will the area of the segment cut off by AM from the given curve, be always equal to the segment cut off by AM′ from the curve, which is the locus of M′. Required the demonstration.

QUESTION XVII. (170)—*By S. of G.*

Given the base, the vertical angle, and the angle subtended by the base at a point which divides the perpendicular of the triangle in a given ratio. Required to construct the triangle geometrically.

QUESTION XVIII. (171)—*By Mr. William Lenhart.*

In a triangle whose base is 50 and perpendicular from the vertical angle on the base 48; it is required to find a point from which if two straight lines be drawn to the angles at the base, the sum of the squares of the sides of the triangle may be to the sum of the squares of the two straight lines thus drawn, as 5 to 2.

QUESTION XIX. (172.)—*By Mr. M. O'Shannessy, A.M.*

A spherical body of a given magnitude is projected in a given direction with a given impetus; required the equation of the curve that will constantly touch it in every point of its trajectory, both curves being in the same plane.

QUESTION XX. (173)—*By Professor Strong, Rutgers College.*

Determine the equation of the curve which is such that its length varies as its radius of curvature corresponding to its variable length.

QUESTION XXI. (174)—*By the same.*

Let any number of parabolas whatever have the same vertex; it is required to find the nature of the curve which, cutting all the parabolas, shall cut off equal arcs, reckoned from the common vertex to the required curve.

QUESTION XXII. (175)—*By Philotechnus.*

Two simple pendulums, the length of which are a and b inches, are suspended to the same point and have their oscillating points connected by an inflexible line or wire, equal c inches; find the tension of this line when the compound pendulum moves by the action of gravity.

THE MATHEMATICAL DIARY:

CONTAINING

NEW RESEARCHES AND IMPROVEMENTS

IN THE

MATHEMATICS:

WITH

COLLECTIONS OF QUESTIONS,

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INGENIOUS CORRESPONDENTS.

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THE

MATHEMATICAL DIARY.

NO. X.

Article XX.

DR. ANDERSON'S *Solution to the Prize Question continued from page 37.*

IN the solution of the question as proposed, I have assumed that the plate will continue always nearly parallel to its original position. That this will necessarily follow from the hypothesis of rolling along the variable projection will appear from the results of the following analysis, which by means of an integration somewhat remarkable, conducts us to a complete solution of the general problem free from all restrictions on the values of the arbitrary constants, and applicable therefore to every possible position and motion of the plate.

When one surface moves on another, each of the two surfaces will bear the traces of a curve described at the same time by the common point of contact. If the rolling be unaccompanied by sliding, the elements of these two curves will at the point of contact be equal in *length*, and will have the same *direction*; and if the remaining condition of the question be included, the *curvature* also of the trace on the horizontal plane will be equal to the curvature of the projection of the circumference of the plate. I proceed to the determination of the general equations, and will then regard the plate, first as *not subject*, and afterwards as *subject*, to this additional condition.

The values of the velocities of the centre of gravity derived from the general equations of rolling motion given by me in the last Number of the Diary, are at present

$$dx' = -\alpha c\, d\theta - \alpha r \cos\psi,$$
$$dy' = -\alpha c'\, d\theta + \alpha r \sin\psi,$$
$$dz' = -\alpha c''\, d\theta\,;$$

φ, θ, ψ being as in Laplace, the rest as in Lagrange.

By means of these expressions the Dynamical Equation will become

$$o = \begin{cases} [d(A\sin^2\theta\, d\psi - Br\cos\theta) - \mu\alpha^2\cos\theta(dr + \sin\theta\, d\theta\, d\psi)]\,\delta\psi \\ [d.Br + \mu\alpha^2 (dr + \sin\theta\, d\theta\, d\psi)]\,\delta\varphi \\ [d.A'd\theta - \sin\theta\, d\psi(A\cos^2\theta + B'r) + \mu\alpha g\cos\theta]\,\delta\theta \end{cases}$$

where A and B denote the principal moments of inertia referred to the centre, A' and B' the same moments referred to a point in the circumference, the symbol dt being at the same time omitted to save room.

Let us first suppose that there exists no condition but that of perfect rolling. The three variations will in this case be arbitrary, and we shall therefore have

$$o = B'dr + \mu\alpha^2 \sin\theta\, d\theta\, d\psi,$$
$$o = A\, d.\sin^2\theta\, d\psi + Br\sin\theta\, d\theta,$$
$$o = A'd^2\theta - B'r\sin\theta\, d\psi - A\sin\theta\cos\theta\, d\psi^2 + \mu\alpha g\cos\theta.$$

These are the general equations of motion. If we confine ourselves to the consideration of the cases in which the plate remains always nearly upright, the complement of θ will be a very small variable. In the last equation, all the terms but the second will now be very small, and therefore the second term must itself be very small. As $\sin\theta = 1$ nearly, this cannot take place unless either r or else $d\psi$ be very small. This alternative gives rise to two distinct problems. In one of these, the plate *spins* swiftly round, supported by a point of contact which changes very slowly its position on the rim; in the other, the plate *rolls* rapidly forward on the horizontal plane, changing at the same time very slowly its course along the plane. I shall consider the *rolling plate* first. The above equations furnish, for this case,

$$r = r'$$
$$\frac{d\psi}{dt} = \psi_{,} + \frac{Br'}{A}(\vartheta - \vartheta')$$

$$\frac{d^2\vartheta}{dt^2}+\frac{BB'r'^2-A\mu a g}{AA'}\vartheta=\frac{B'r'}{A'}\left(\frac{Br'\vartheta'-A\psi_{,}}{A}\right);$$

where ϑ denotes the complement of θ, and ϑ', r', $\psi_{,}$, the values of ϑ, r and $\frac{d\psi}{dt}$, when $t=o$; the time t being counted from the instant when $\frac{d\vartheta}{dt}=o$.

The last equation gives by integration, (e^2 denoting the coefficient of the second term)

$$\vartheta-\vartheta'=\frac{\mu a g\vartheta'-B'r'\psi_{,}}{A'e^2}(1-\cos et).$$

If the plate was forced primitively from a state of rest, by an impulse in the direction of its place, the term containing $\psi_{,}$ will disappear. The plate will then sink from its primitive inclination, and though it will return to it at stated intervals, it will never be more erect than it was when first impelled. The mean orbit of the point of contact will be a very large circle whose radius will be subject to an inequality proportional to the cosine of et. The actual orbit therefore will be a species of harmonic curve, of which the mean orbit will be the base. If we consider the plate as the equator of an imaginary globe, its motion will bear a strong resemblance to the motion of the terrestrial equator. The longitude of the equinoxes will be affected by two inequalities; one secular proportional to the time and equal to $\frac{B\mu a g r'\vartheta'}{AA'e^2}t$, the other periodical and represented by $\frac{B\mu a g r'\vartheta'}{AA'e^3}\sin et$.

If the rotation of the plate round its axis be considered as direct, the mean motion of the equinoxes will be uniformly retrograde; and as the mean obliquity of the plate will be constant, the phenomena of precession and nutation will evidently be the consequence. The mean poles of the plate will describe lesser circles parallel to the ecliptic and very near to it, while the real poles will describe around the mean poles small ellipses, whose semi-axes will be equal to $\frac{\mu a g\vartheta'}{A'e^2}$ and $\frac{B\mu a g r'\vartheta'}{AA'e^3}$.

The only method of enabling the plate to elevate itself above its primitive obliquity is to direct the initial impulse so as to make a small angle with the plate. If this impulse be so adjusted that $A\psi_{,} = B\vartheta' r'$, the mean position of the plate will be vertical, and the mean path of the point of contact a straight line. We shall presently see that this will always be the case when the last condition of the question is included.

In the problem of the *spinning plate*, the general equations will be found to furnish the following solution,

$$\frac{d\psi}{dt} = \psi_{,}$$

$$\vartheta - \vartheta' = \frac{\mu a g \vartheta' - A\psi_{,}^2\vartheta' - B' r' \psi_{,}}{A' e'^2}(1 - \cos e' t),$$

$$r - r' = \frac{\mu a^2 \psi_{,}}{B'}(\vartheta - \vartheta');$$

where $A' e'^2 = A'\psi_{,}^2 - \mu a g$.

Thus it appears that in spinning a circular plate placed not exactly but nearly upright on a horizontal plane, if we wish it to retain permanently its nearly upright position, it must be first put in motion by a force capable of producing an angular velocity round the vertical at least greater than $\sqrt{\dfrac{\mu a g}{A + \mu a^2}}$. This velocity round the vertical will then continue constant, but the velocity round an axis perpendicular to the plate will continually vary, and in some cases may change from direct to retrograde, but will remain always very small. The point of contact will gradually work its way around the rim of the plate with an unequal and very slow motion, and the plate at the same time that it spins swiftly round the vertical and revolves very slowly and irregularly on its axis, will oscillate on both sides of an invariable mean obliquity equal to $\dfrac{\mu a^2 \psi_{,}^2 \vartheta' - B' r' \psi_{,}}{A'}$.

Secondly. If we now include the condition of rolling along the variable projection, we must first express the equality of the elements of the two traces in length and in direction by the two equations

$$dx = -a\,d\phi \cos\psi,$$
$$dy = a\,d\phi \sin\psi.$$

These equations exist also in the preceding cases. In the present instance the curvature of the horizontal trace is moreover equal to the curvature of the horizontal projection of the other. This is expressed by the equation

$$\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y-dyd^2x}=\frac{a}{\cos\theta}.$$

Substituting the above values and reducing, we get

$$d\psi-\cos\theta d\phi=0.$$

The first member of this equation represents in general the elementary rotation dN round a vertical axis. This rotation therefore is reduced to zero by the last condition of the problem, whatever be the inclination of the plate. The variations $\delta\psi$ and $\delta\phi$ being now no longer arbitrary, but connected by the above equation, the substitution of $\cos\theta.\delta\varphi$ in place of $\delta\psi$ will furnish us with the following two equations,

$$0=\cos\theta d(A\sin^2\theta d\psi-Br\cos\theta)+\mu a^2\sin^2\theta(dr+\sin\theta d\theta d\psi)$$
$$0=A'd^2\theta-\sin\theta d\psi(A\cos\theta d\psi+B'r)+\mu ag\cos\theta.$$

Two integrals of the first order and in finite terms may be obtained from these equations. By combining the equation $d\psi=\cos\theta d\varphi$ with the well known equation $r=d\varphi-\cos\theta d\psi$, we obtain

$$d\psi=\frac{r\cos\theta}{\sin^2\theta},$$

by means of which the first of the above equations is transformed to

$$(A-B)\cos\theta d(r\cos\theta)+\mu a^2\sin\theta d(r\sin\theta)+Bdr=0,$$

and by further reduction to

$$[A\cos^2\theta+(B+\mu a^2)\sin^2\theta]dr=[A-(B+\mu a^2)]r\sin\theta\cos\theta d\theta.$$

Whence

$$\frac{dr}{r}=\frac{(A-B')\sin\theta\cos\theta d\theta}{A\cos^2\theta+B'\sin^2\theta}.$$

Integrating by logarithms, and then passing to numbers, we obtain

$$(A\cos^2\theta+B'\sin^2\theta)r^2=\text{const.}=k^2.$$

By means of this integral and the equation $d\psi=\cos\theta d\phi$ we may now obtain from the second of the equations above

referred to another integral in finite terms. This last equation may be first transformed to

$$A'\frac{d^2\theta}{dt^2} - r^2(A\cos^2\theta + B'\sin^2\theta)\frac{\cos\theta}{\sin^3\theta} + \mu a g\cos\theta = o$$

and then by virtue of the preceding integral to

$$A'\frac{d^2\theta}{dt^2} - \frac{k^2\cos\theta}{\sin^3\theta} + \mu a g\cos\theta = o;$$

the integral of which is

$$dt = d\theta\sin\theta\sqrt{\frac{A'}{l\sin^2\theta - 2\mu a g\sin^3\theta - k^2}};$$

l being a second arbitrary constant.

From the preceding equations the position of the plate at any given instant of time may be determined by the method of quadratures whatever be the values of the eight arbitrary constants which belong to this form of the question.

If we now suppose the plate to continue during the motion very nearly upright, the first of the two integrals above given will furnish $B'r^2 = k^2$, and therefore r constant. The other equations give us

$$\frac{d^2\vartheta}{dt^2} + \frac{B'r^2 - \mu a g}{A'}\vartheta = o$$

$$\frac{d\psi}{dt} = r\vartheta,$$

$$\vartheta = \vartheta'\cos et,$$

$$\psi = \psi_{,}\sin et.$$

Whence it is evident that the inequalities of the inclination and the node-line are periodical, and confined within very narrow limits. The mean position of the plate must be vertical and the mean path of the point of contact and centre of gravity must be straight lines. All these results, it will be seen, differ only in notation from those which I gave in the last number of the Diary.

Solution, to the Prize Question in No. VIII.

By Professor Strong.

Let R denote the radius and M the mass of the plate regarded as a physical plane, and s denote the line described by the point of contact on the horizontal plane in the time t,

ϕ = the angle made by a radius of the plate fixed in its plane with the tangent at the extremity of s at the time t from the origin of the motion. (This angle denoting the rotation of the plate round its axis when counted from the tangent). Also suppose that ψ denotes the angle made by the same tangent with a tangent at the origin of s which is taken for the axis of x (the axis of y being drawn in the horizontal plane at right angles to the tangent through the point of contact) and θ = the inclination of the plate to the horizontal plane at the time t. Now since the plate rolls without sliding in any direction, we may consider the plate as simultaneously revolving about its diameter which passes through the point of contact, about an axis which passes through its point of contact parallel to its axis and about the tangent at the extremity of s, (as it passes into the following element ds of s) with the respective angular velocities $\frac{d\psi \sin\theta}{dt}, \frac{d\phi - d\psi\cos\theta}{dt}, \frac{d\theta}{dt}$. Put $\text{A} = \frac{\text{MR}^2}{4}$ $\text{C} = 2\text{A} + \text{MR}^2 = \frac{6\text{MR}^2}{4}$, $\text{B} = \text{A} + \text{MR}^2 = \frac{5\text{MR}^2}{4}$, which are the respective moments of inertiæ about the aforesaid axes. Hence by known principles $\frac{\text{C}(d\phi - d\psi\cos\theta)^2 + \text{A}d\psi^2\sin^2\theta + \text{B}d\theta^2}{dt^2}$ $= \frac{\text{MR}^2}{4} \times \frac{6(d\phi - d\psi\cos\theta)^2 + d\psi^2\sin^2\theta + 5d\theta^2}{dt^2}$ = the whole living force of the rolling plate.

Let T denote have the living force, g = gravity, $\text{BM}g = c'$, and $c'\sin\theta = \text{V}$. Then (see La Grange, Mec. Anal. vol. 1st, page 313, art. 10th), I have $d.\frac{\delta\text{T}}{\delta d\phi} = 0$ (a); $d.\frac{\delta\text{T}}{\delta d\psi} = 0$ (b); $d.\frac{\delta\text{T}}{\delta d\theta} - \frac{\delta\text{T}}{\delta\theta} + \frac{\delta\text{V}}{\delta\theta} = 0$ (c): (a) gives $d.\left(\frac{d\phi - d\psi\cos\theta}{dt}\right) = dp = 0$ $\therefore$ $\frac{d\phi - d\psi\cos\theta}{dt} = p = \text{const.}$ (1) also (b) gives $d.(\text{A}\sin^2\theta\frac{d\psi}{dt} - cp\cos\theta) = 0$ $\therefore$ $\text{A}\sin\theta^2\frac{d\psi}{dt} - cp\cos\theta = \text{const.}$ but at the origin $\frac{d\psi}{dt} = 0$ $\therefore$ $-cp$

$\cos\theta' =$ const. ($\theta' =$ the initial value of θ) ; hence I have A $\sin^2\theta \frac{d\psi}{dt} - cp \cos\theta = -cp \cos\theta'$ (2) and $\frac{d\psi}{dt} = cp \frac{(\cos\theta - \cos\theta')}{\text{A}\sin^2\theta}$ (3). Again (c) gives $\frac{\text{B}d^2\theta}{dt^2} - cp \sin\theta \frac{d\psi}{dt} -$ A $\sin\theta\cos\theta \frac{d\psi^2}{dt^2} + \text{c}'\cos\theta = 0$, ($f$) ; the value of $\frac{d\psi}{dt}$ substituted in this reduces it to $\frac{\text{B}d^2\theta}{dt^2} + \frac{(c^2p^2(\cos\theta' - \cos\theta)}{\text{A}\sin\theta} - \frac{c^2p^2\cos\theta \times (\cos\theta' - \cos\theta)^2}{\text{A}\sin^3\theta} + \text{c}'\cos\theta = 0$ (4) ; this multiplied by $d\theta$ integrated and corrected by making $\frac{d\theta}{dt} = 0$, and $\theta = \theta'$ (at the origin) gives B $\frac{d\theta^2}{dt^2} = 2\text{c}'(\sin\theta' - \sin\theta) - \frac{c^2p^2(\cos\theta' - \cos\theta)^2}{\text{A}\sin^2\theta}$ (5). By putting in (3) $\frac{6\text{MR}^2}{4}$ for c and $\frac{\text{MR}^2}{4}$ for A, it becomes $\frac{d\psi}{dt} = \frac{6p(\cos\theta - \cos\theta')}{\sin^2\theta}$ the same as (3) of my former solution, and by putting in (f) for B and c their values it becomes $\frac{5}{4}\frac{d^2\theta}{dt^2} - \frac{\sin\theta}{4}.\frac{d\psi}{dt}\left(\frac{d\psi}{dt}\cos\theta + 6p\right) + \frac{g}{\text{R}}\cos\theta = 0$ which is (4) of my former solution, and p is the same in this as in that solution ; and I would remark that ϕ, ψ, θ, p, M, R are the same as in that solution, but I have used A, B, C, &c. to show the agreement of my former solution in its general results with those given by Mr. Nulty, in his solution inserted in the 9th No. of the Diary. If in (4) we change sin. into cos. and cos. into sin. before θ, θ', in the several terms, and $\frac{d^2\theta}{dt^2}$ into $-\frac{d^2\theta}{dt^2}$ the angle, θ thence resulting must be reckoned from the vertical. Making these changes there results $\frac{d^2\theta}{dt^2} + \frac{c^2p^2 - \text{AC}'}{\text{AB}}\theta - \frac{c^2p^2}{\text{AB}}\theta' = 0$, (6)

supposing that θ and θ' are so small that quantities of the first order of $\theta\ \theta'$ are only retained. (6) Integrated and corrected by making $\theta = \theta'$ at the origin gives $\theta = \frac{1-n\cos\kappa t}{1-n}\times\theta'$ (7), $\left(n = \frac{\text{A}c'}{c^2p^2}\right)\kappa = \sqrt{\left(\frac{c^2p^2-\text{A}c'}{\text{AB}}\right)}$, the limits of θ being θ' and $\frac{1+n}{1-n}\theta'$. Also $\psi = \frac{m\theta'}{\kappa(1-n)}$ $\times(\kappa t - \sin\kappa t)$ (8). $\left(m = \sqrt{\frac{(c'n)}{\text{A}}}\right)$ also $\phi = pt$ (9), these results are the same as those found in my former solution, whence every thing else can be found after the same manner as in that solution. The results here obtained do not hold except $\frac{1+n}{1-n}$ is a finite quantity, and θ' an indefinitely small angle, but in those cases which make an exception to the process here used the solution may be obtained by the aid of (5) by the usual methods of approximation. It appears also from what has been done above, that the angle which the plate makes with the vertical cannot be less than the initial angle; but this may be shown by (5) in a general manner, for at the highest points $\frac{d\theta}{dt} = 0$ in which case (5) gives us $2c'x(\sin\theta' - \sin\theta) = \frac{c^2p^2(\cos\theta' - \cos\theta)^2}{\text{A}\sin^2\theta}$ the right hand side of this equation is of necessity always positive $\therefore$ the left hand side is positive also; and hence θ never can exceed θ'.

I will add the following remarks to what has been said above for the purpose of showing the truth of the fundamental equations which I have obtained. To the end proposed I observe that since the plate must roll without any sliding, it is evident that the resutant of all the forces which act upon the plate at its point of contact including friction and the reaction of the plane, must pass through the point of contact and through the centre of the plate; and that the said resultant must equal the pressure of the plane estimated in the direction of its radius drawn to the point

of contact. For should the forces on one side be greater than the opposing forces on the opposite side, the plate must of necessity slide, which would be against the hypothesis. Hence it is evident that the forces estimated in the direction of the plane of the plate must be considered as having no effect upon its motion. Indeed it is evident that the progressive motion of the centre of the plate, estimated in the direction of its plane, carries the point of contact forward, just as much as the rotation around its axis carries it backward as it passes over the successive elements of the curve which the point of contact describes on the horizontal plane; so that no friction in the direction of the elements can exist. Or to show the same thing differently, we may suppose the curve traced out on the horizontal plane to be a groove perfectly smooth on the inside, and that the plate before projection revolves about its axis so that the velocity of any point of its circumference around its centre equals the velocity with which its centre is projected in the direction of a line parallel to the initial element of the groove; and the motion will be the same as in the question. From these considerations (1) results the same as before. Again, since the resultant passes through the point of contact and centre of the plate it is manifest that it cannot affect the horizontal areas traced by the respective particles around a vertical to the horizontal plane drawn through the point of contact as the plate, moves. From this consideration and that the point of contact is momentarily at rest in consequence of the rotation and the progressive motion of the centre, there results by the well known principle of areas, (2) the same as before. Again since the resultant of the resistances passes through the point of contact and centre of the plate, the principle of the preservation of living forces must have place which gives, (5) as above, (1), (2), and (5) being found every thing else results as before. By considering the plate as a polygon of an infinite number of sides, and supposing that it moves in such a manner as to have its sides successively applied to the plane without any sliding, the same conclusions may be obtained.

I will now suppose that the motion of the plate is to be subjected to the partial differential equation $\frac{d\phi}{d\psi} = \frac{1}{n \cos \theta}$ (*a*), and that $\cos \theta$ is always a very small quantity. Then by La Grange's process (before cited,) I have $\left(d.\frac{\partial T}{\partial d\phi}\right)$ $\partial\phi + \left(d.\frac{\partial T}{\partial d\psi}\right)\partial\psi + \left(d.\frac{\partial T}{\partial d\theta} - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta}\right)\partial\theta = 0$ (1), but in (*a*) we may manifestly change $\frac{d\phi}{d\psi}$ into $\frac{\partial\phi}{\partial\psi}$, which gives $\partial\psi$ $= n \cos \theta\partial\phi$, substitute this in (1), and put the coefficients of $\partial\phi \,.\, \partial\theta$ separately $= 0$, and we have $d.\frac{\partial T}{\partial d\phi} + n \cos \theta$ $d.\frac{\partial T}{\partial d\psi} = 0$ (2) $d.\frac{\partial T}{\partial d\theta} - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta} = 0$ (3), but since $\cos \theta$ is supposed to be an indefinitely small quantity of the first order, it is manifest that the second term of (2) is of the second order of minuteness relatively to the first term, and is hence to be neglected; hence (2) becomes $d.\frac{\partial T}{\partial d\phi} = 0 = cd\frac{(d\phi - d\psi \cos \theta)}{dt^2} = c\frac{d^2\phi}{dt^2} = 0$, because $d\psi \cos \theta$ is a quantity of the second order relatively to $d\phi$, hence $\frac{d\phi}{dt} = m =$ const. Also (3) gives by omitting indefinitely small quantities after the differentiation as above $\text{B}\frac{d^2\theta}{dt^2} - cm \sin \theta\frac{d\psi}{dt} + g \cos \theta = 0 = \text{B}\frac{d^2\theta}{dt^2} - nc$ $m^2 \sin \theta \cos \theta + g \cos \theta$ (since (*a*) gives $\frac{d\psi}{dt} = n \cos \theta m$) which gives $\frac{d^2 u}{dt^2} + \frac{ncm^2 - g}{\text{B}}u = 0$ (4) ($u = \frac{p}{2} - \theta$; $p =$ the semicircumference rad. (1), the angle u being reckoned from the vertical), (4) integrated and corrected by supposing u' to be the initial value of u, and by putting $\frac{ncm^2 - g}{\text{B}}$ $= e^2$, gives $u = u' \cos et$ $\therefore \psi = \frac{nu'm}{e} \sin et$ and $\phi =$

mt. Also $x = \text{R}\phi$ nearly $= \text{R}mt$ and $y = \dfrac{2\text{R}nm^2u'}{e^2} \sin^2 \dfrac{et}{2}$ and $x' = \text{R}mt$; $y' = \text{R}u' \cos et + \dfrac{2\text{R}nm^2u'}{e^2} \sin^2 \dfrac{et}{2}$; $z' = \text{R}$ x, y being the coordinates which define the motion of the point of contact, and x', y', z', those which define the motion of the centre of gravity of the plate, their common origin being at the point of contact of the plate with the horizontal plane at the origin of the motion. The direction of the axis of x being that of a tangent to the described curve at its origin. If $n = 1$, $e^2 = \dfrac{6m^2\text{R}-4g}{5\text{R}}$, and the equation of condition is $d\phi = \dfrac{d\psi}{\cos\theta}$ which arises from the equality of the radii of curvature of the described curve, and the variable projection at their point of contact, so that the plate describes its variable projection in this case, and the results which I have obtained, are the same as those found by Dr. A. in his solution of the same case of the general question by changing u, u' into $\theta. \theta'$, and ψ into $-\phi$, R into a, and by correcting his value of y, so that it shall $= 0$, at the origin. For where I use u, u', he uses θ, θ', &c. for the same things. Again, if $n = 2$, $e^2 = \dfrac{12m^2\text{R}-4g}{5\text{R}}$, which is the value of e^2, found by Dr. A., which he says, gives the general solution of the question, when the plate is not supposed to describe its variable projection, but to take the course which it will naturally take, according to the laws of motion, so as to satisfy the remaining conditions. But I have to remark that the equation of condition, when $n = 2$, is $d\phi = \dfrac{d\psi}{2\cos\theta}$, so that Dr. A. has in his solution virtually by his process restricted the motion of the plate, so as to satisfy the 2d equation of condition. What has been done above, shows the reason of the disagreement between the results found by Mr. N., and Dr. A., in their solutions of this question in the general case. For they both obtained the equation $dm = 0 = dr$, (in Mr. N's notation), which requires that the tangental force in the direction of the curve described by the point of

contact on the horizontal plane should $= 0$, so that they do not disagree in this respect. But they disagree in this, that Dr. A. subjects the motion of the plate to the equation of condition $d\varphi = \frac{d\psi}{2\cos\theta}$ which gives $\frac{d\psi}{dt}\sin\theta = 2\cos\theta \sin\theta\frac{d\varphi}{dt} = m\sin 2\theta$, but $\frac{d\psi}{dt}\sin\theta$ denotes the angular velocity of the plate around its diameter which passes through its changing point of contact; consequently, Dr. A. subjected the plate to move in such a manner that its angular velocity about its diameter should be denoted by $m \sin 2\theta$, whereas Mr. N. did not suppose any such velocity; but supposed the said velocity to be such as the laws of motion require it to be, when $\delta\varphi$, $\delta\psi$, $\delta\theta$, are considered as independent variations. The general solution which I have given, supposing that n is any finite positive number, does manifestly give as just a solution of the general case as that proposed by Dr. A. Thus if $n = 3$, $e^2 = \frac{18m^2R - 4g}{5R}$, &c. would answer equally as well. Many other remarks might be made, but they are so obvious, that I need say no more about them, and I shall conclude by observing that Mr. N's solution applies to the general case, but that the other does not. Perhaps I owe an apology to Dr. A., for making the above observations on his solution. I assure him that I have not made them from any desire to injure him. My only object has been to explain a subject which has been thought by some of the ablest mathematicians in this country to be involved in insuperable difficulties.

ON PERFECT ROLLING MOTION.

By Eugenius Nulty, Philadelphia.

If a solid body roll in contact with a surface, and its elements be referred to principal axes a, b, c, half the living force, diminished by the sum of the actions of gravity will be

$$F=\frac{1}{2}(Ap^2+Bq^2+Cr^2)+\frac{m}{2}\{(c'q-b'r)^2+(a'r-c'p)^2+(b'p-a'q)^2\}-mgz'.$$

This formula is involved in the general expression found from obvious principles, in the preceding Number of the Diary. In its present form, the known values

$$cq-br,\ ar-cp,\ bp-aq,\ \&c.$$

in the directions of the principle axes are introduced instead of the general components α, β, γ, &c. and the moments of inertia are represented as usual by A, B, C: it becomes directly applicable to a body rolling on a surface, or revolving round a fixed point. In the first case, the co-ordinates a', b', c' of the point of contact will be variable; in the second, these co-ordinates may be regarded as belonging to the centre of gravity relatively to the fixed point, and they will be constant. These considerations immediately lead, on taking the variation of F, to a remarkable expression for the difference between the developement of this formula in case of a fixed point, and that corresponding to perfect rolling motion. It may be presented in the following simple form by assuming

$$\varrho^2=a'^2+b'^2+c'^2 \text{ and } \varrho'=a'p+b'q+c'r:$$

$$\delta F'=\frac{m}{dt}\left\{\left(\frac{1}{2}pd.\varrho^2-\varrho'da'\right)\delta P+\left(\frac{1}{2}qd.\varrho^2-\varrho'db'\right)\delta Q+\left(\frac{1}{2}rd.\varrho^2-\varrho'dc'\right)\delta R\right\}.$$

In this expression ϱ and ϱ' are independent of the directions of the axes of a, b, c, but if the body be referred to axes x, y, z, of which the two first are in a horizontal plane, the differentials $\frac{dx'}{dt}$, &c. of the new co-ordinates of the point of contact, must be diminished by the known components $z'm-y'n$, &c.

Let now the rolling body be a solid of revolution with respect to the principal axis of c, and let it be projected on a horizontal plane in the direction of its equator supposed to be nearly vertical. The co-ordinate c' and the quantities p, q involving φ, ψ, ω will then be small; ω will be nearly

equal to a right angle; ϱ may be taken for the radius of the equator; and if we put μ for the semi-parameter of the generating curve regarded as of the second degree at the changing point of contact (a', b', c'), we shall have by Geometry, the approximate values

$$a' = \varrho \sin \phi,\ b' = \varrho \cos \varphi,\ c' = \mu \cos \omega,$$

and the vertical co-ordinate $z' = \varrho - \frac{1}{2}\left(\varrho - \mu\right) \cos \omega^2.$ By virtue of these and the known values of p and q, we have $\varrho' = \varrho \sin \omega \psi' + \mu r \cos \omega$, in which $\psi' = \frac{d\psi}{dt}$; wherefore by substitution

$$\delta F' = -\varrho^2 m.\sin \omega.\phi'\psi'(\cos \varphi \delta P - \sin \phi \delta Q),$$

which reduced to axes x, y in the horizontal plane, becomes

$$\delta F' = -\varrho^2 m.\sin\omega.\phi'\psi'(\cos \psi \delta L + \sin \psi \delta M),$$

and with respect to a momentary axis passing through the point of contact, it takes the simple form

$$\delta F' = -\varrho^2 m.\sin\omega.\phi'\psi' \delta\omega.$$

These are the different expressions by virtue of which the equation corresponding to a body revolving round a fixed point, are rendered applicable to perfect rolling motion. They are extremely simple; their accuracy may be easily tested by a consideration of the moments of $\varrho \sin \omega \psi'$ and the element $\varrho d\varpi$; and they immediately lead to the remarkable expressions

$$dx + dx' = \varrho d\varphi.\cos \psi,\ dy + dy' = \varrho\ d\varphi.\sin \psi,$$

which will make known the curve described by the point of contact (x', y', z'), and also the projection of the path traced by the centre of gravity (x, y, z).

Let us determine the equations of motion. The formula F becomes in the present case by assuming

$$C' = C + \varrho^2 m,\ C'' = mg\ (\varrho - \mu),$$

$$F = \frac{1}{2}\left\{ A\,(p^2 + q^2) + C' r^2 \right\} + \frac{m}{2}\left\{ (b'p - a'q)^2 - 2rc'(a'p + b'q) \right\} - \varrho m g + \frac{1}{2} C'' \cos \omega^2,$$

the development of which in terms multiplied by δP, &c. δL, &c. and $\delta\phi$, $\delta\psi$, $\delta\omega$ may be found in the M. Analytique. We may obtain it by means of the formula given in the Diary. We may neglect $c'(a'p+b'q)$, which produces terms of the second order, and put $a'=0$, which the form of the body renders admissible. We shall then find on the evanescence of ϕ after differentiation, and on the addition of $\delta F'$,

$$\left(C'\frac{dr}{dt}-\varrho^2 m.\sin\omega.\psi'\omega'\right)\delta\phi+\left\{\frac{d\psi}{dt}\left(A\sin\omega^2.\psi'+C'r\cos\omega\right)\right.$$
$$\left.+\varrho^2 m.\sin\omega.\phi'\omega'\right\}\delta\psi+\left(A'\frac{d^2\omega}{dt^2}+C'r\sin\omega.\psi'-\right.$$
$$\left.A\sin\omega.\cos\omega.\psi'^2+C''\cos\omega\right)\delta\omega=0.$$

The small oscillations of a variety of solids may be determined from this expression. It will apply to an Elliptic Spindle, and its symmetrical zones, to a spheroid, to a hyperbolic or parabolic spindle, &c.; and if we suppose $\mu=0$ and therefore $c''=\varrho m g$, it will present the complete development corresponding to a circle projected on the horizontal plane, consistently with perfect rolling motion. In case of the small oscillations of a circle, and of the solids mentioned, we may put $\frac{\pi}{2}-\omega=\theta$, a small quantity, and $\sin\omega=1$, $\cos\omega=\theta$. The preceding expression will then become

$$C'\frac{d\phi'}{dt}\delta\phi+\left(A\frac{d\psi'}{dt}+C\phi'\theta'\right)\delta\psi+\left(A'\frac{d\theta'}{dt}-C'\phi'\psi'-C''\theta\right)$$
$$\delta\omega=0. \qquad \text{(A)}$$

It may not be unworthy of notice, that the angles ψ and θ in this expression may be changed into the cosines ξ''', ζ''', which correspond to the angles formed by the axis of c and the axes x and z, and that the variations δL, δM, δN may be substituted for $\delta\omega$, $\delta\phi$, $\delta\psi$. By this transformation we shall have

$$\left\{A'\frac{d^2\zeta'''}{dt}-C'm\frac{d\xi}{dt}+C''\left(\frac{\pi}{2}-\zeta'''\right)\right\}\delta L-C'\frac{dm}{dt}\delta M-$$
$$\left(A\frac{d^2\xi}{dt^2}+Cm\frac{d\zeta'''}{dt}\right)\delta N=0,$$

which may be also determined from a formula in the last page but one of the M. Analytique, with the additional terms in the second value of $\delta F'$.

If we now consider the variations $\delta\phi$, $\delta\psi$, $\delta\omega$ as independent of each other, there will result

$$\frac{d\phi'}{dt}=0,\ A\frac{d\psi'}{dt}+C\phi'\theta'=0,\ A'\frac{d\theta'}{dt}-C'\phi'\psi'-C''\theta=0.$$

From the first of these equations we obtain $\phi'=h$, $\phi=ht$. The angular velocity of the body round the principal axis c is therefore constant; and by substituting for ϕ', the second and third equations become

$$A\frac{d\psi'}{dt}+Ch\theta'=0,\ A'\frac{d\theta'}{dt}-C'h\psi'-C''\theta=0.$$

In order to integrate these equations, assume $\theta'=0$ at the origin of motion, and let ψ' and θ vanish together. We shall then have $\psi'=-\frac{Ch}{A}\theta$, by virtue of which the third equation becomes

$$\frac{d^2\theta}{dt}+\frac{CC'-AC''}{AA'}\theta=0,$$

of which the integral, by putting the coefficient of θ equal to k^2, is well known to be $\theta=k'\cos kt$. The body will therefore oscillate between the limits k' and $-k'$, and the time of describing the arc $2k'$ will be $t=\frac{\pi}{k}$. The value of θ just obtained being substituted in $\psi'=-\frac{Ch\theta}{A}$, there will result $\psi=-\frac{Chk'}{Ak}\sin kt$, which will determine the deviation of the equator from a right line in its primitive direction relatively to the horizontal plane. The extreme value of this angle is $\frac{Chk'}{Ak}$; it is attained when $\sin kt=\pm 1$.

These values of φ, ψ, ω will make known the motions of the body at any instant t. In conjunction with the coordinates

$$a' = \rho \sin\phi, \; b' = \rho \cos\phi; \; c' = \rho\theta,$$

they enable us to find the components,

$$\alpha' = -b'r, \; \beta' = a'r, \; \gamma' = b'p - a'q,$$

which, multiplied by the known values of the cosines ξ', ξ'', &c. of the angles formed by the axes a, b, c and x, y, z, will give the velocities of the centre gravity, and therefore the co-ordinates of the curve described by this point in space. The differences between these co-ordinates and those of the point of contact relatively to the axis x, y, z, and the centre of gravity will evidently determine the path of the body on the horizontal plane. These views are general. In the present case we have $\cos\psi = 1$, $\sin\psi = \psi$ and by formula already given

$$dx + dx' = \rho\phi' dt, \; dy + dy' = \rho\phi'\psi dt$$

by virtue of which and the values of ϕ' and ψ, the co-ordinates of the curve described on the plane are

$$x + x' = \rho h t, \; y + y' = -\rho \frac{Ch^2k'}{Ak^2} \cos kt.$$

and since $x' = 0$ and $y' = \rho\theta$, the co-ordinates of the projection of the path of the centre of gravity are

$$x = \rho h t, \; y = -\rho k' \left(\frac{Ch^2 - Ak^2}{Ak^2}\right) \cos kt.$$

Conceive the circle of declination passing through the point (a', b', c') to be orthogonally projected on the horizontal plane. The diameters of the ellipse thus formed may be considered as equal to ρ and $\rho \cos\omega = \rho\theta$. The radius of curvature at the extremity of $\rho\theta$ will therefore be $\frac{\rho}{\theta}$, and equating the elements $\rho \frac{d\psi}{\theta}$ and $\rho d\phi$, we shall have at the point of contact $d\psi - \theta d\phi = 0$. This expression is equivalent to the imperfect differential $ndt = 0$. It cannot be integrated in consequence of the variability of θ. If we admit the passage from it to the expression $\delta\psi - \theta\delta\phi = 0$, which would result from the incomplete variation, $\delta N = 0$, and introduce both expressions into the formula (A), or from that formed from the equations (2), (3), (4) given in the Diary, respectively multiplied by $\delta\phi$, $\delta\psi$, $\delta\omega$, we shall obtain for the co-efficient of θ in the re-

sulting equation of oscillation $k^2 = \frac{C'h - C''}{A'}$. A similar substitution made in the development corresponding to the free motion of the body would give $k^2 = \frac{Ch - C''}{A}$. But these results, and the consequences deduced from the developement of (1), are hypothetical. The terms which this development requires, in order to render the equations (2) and (3) applicable to perfect rolling motion, depend on the partial variations $\frac{\delta\gamma'}{\delta p}$, $\frac{\delta\gamma'}{\delta q}$ of $\varrho^2 m\gamma'\omega'$ involved in $B\omega'^2$. These with the additional term $\frac{\delta F}{\delta\nu}\delta\nu$, to the formula of variations, will lead immediately to

$$\varrho^2 m. \sin \omega.\omega'(\phi'\delta\psi - \psi'\delta\phi),$$

which renders complete the equations above mentioned; and the rotation of the circle, and the areas described on the horizontal plane will be constant only when the oscillations are small.

The preceding solution embraces general principles, which may be easy applied to every case of perfect rolling motion. I shall therefore conclude with a few observations. The fundamental expression $\delta F'$ becomes simply $-\frac{mc'r}{dt}\left(da'\delta P + db'\delta Q\right)$, when a', b', p, q are small, and the differential of c' insensible. These conditions hold with regard to solids of the second degree put in contact with the element of a horizontal plane, and made to gyrate round a principal diameter supposed nearly vertical. The co-ordinates a' and b' in case of such solids and small oscillations, are well known to be of the form $\mu \sin \phi \sin \omega$, and $\mu' \cos \phi \sin \omega$, and the values of δP and δQ being $\sin \phi \sin \omega\delta\psi + \cos \phi\delta\omega$ and $\cos \phi \sin \omega\delta\psi - \cos \phi\delta\omega$, the preceding expression by neglecting quantities of the second order relatively to the small angle ω, therefore becomes

$$\delta F' = mc'r\left\{(\mu - \mu')\sin\phi\cos\phi.\omega' + (\mu\sin\phi^2 + \mu'\cos\phi^2)\,\phi' \sin\omega.\delta\omega\right\};$$

from which we infer that the equations of motion involving $\delta\varphi$, $\delta\psi$ will require no additional terms; and that the angular velocity and the areas described by the projections of dm on the horizontal plane, will be independent of the conditions of perfect rolling motion. In case of the segments of a spheroid and sphere, the incremental term will be simply $-\mu c' r \sin \omega . \varphi' \delta \omega$, which is analogous to the third value of $\delta F'$ found in case of a solid rolling nearly in the direction of its equator.

Solution to the Prize Question in No. VIII.

By Dr. Adrain.

1. In the present solution, the condition relative to the variable projection is omitted, and the investigation in general whatever be the inclination of the plate to the horizon: the problem therefore as here resolved may be stated as follows. To investigate the motion of a uniform heavy circular plate rolling without sliding on a horizontal plane.

2. This solution is founded on the general principle that every solid in motion may be considered as absolutely free, and acted on not only by any accelerative forces, but also by all the forces arising from the reactions of all the bodies with which it is in contact, to which must be added all the conditions of the motion.

3. When all the forces acting on the moving body are thus introduced, the equations of the motion may be obtained by dividing them into two classes. The first of these classes contains the equations of the motion of the centre of gravity, on the well known principle that this motion is the same as if all the forces were applied in their proper quantity and direction to the mass of the body concentrated in its centre of gravity. In this class there are generally three motions in the directions of three rectangular axes fixed in space.

The second class of motions relates to the rotations of the body about its three principal or natural axes. By means of the forces acting on the body and the forces of inertia we can express the accelerations of the angular velocities of the body about its principal axes; and thus we obtain

the six equations which express all the motions of the solid, both in translation and rotation.

4. It then remains to express all the conditions of the motion as given by the question; and the resulting equations, together with the six equations of motion, are all that are necessary to determine the quantities required in the problem.

Of the conditions which occur in the motion of bodies one of the most obvious is that of rolling, without sliding. This condition is equivalent to the supposition that the particle of the moving body which is in contact with the surface on which the body rolls is at rest if the surface be at rest; and if the surface be in motion, the particle of the moving body in contact with it, must have its velocity the same in quantity and direction with the velocity of that part of the surface on which the body rolls.

In the case of a body rolling on a plane at rest, the equation arising from this principle (besides the equation of contact) are two, shewing that the velocitities of the element in contact are each equal to zero in the directions of two fixed rectangular axes in the plane. Such are the general principles by which we may resolve problems relative to the motion of solid bodies, we shall now apply this theory to the problem of the rolling circle.

5. Let $g =$ gravity, $\rho =$ the radius of the plate, x, y, z, the co-ordinates of its centre of gravity, $x_{,}$ $y_{,}$ those of the curve described on the horizontal plane by the point of contact to the same rectangular axes, the axes of x and y being in the horizontal plane, and that of z, vertical upwards. Let ϕ, ψ, θ be three angles which determine the position of the body and of its principal axes at any instant with respect to its centre of gravity; A, B, C the moments of inertia of the solid about its three principal axes fixed in the body of which the first is determined by ϕ the second by $\frac{\pi}{2}+\phi$, both in the plane of the circle, the third axis, viz. that of z, being at right angles to the plane of the circle: also let p, q, r be the angular velocities of the solid about the axes of C, A, B. Lastly, it is evident that all the re-actions of the plane on the circle may be reduced to three

forces acting at the point of contact, viz. one vertically upwards, and two horizontal, of which one is perpendicular to the tangent of the circle at the point of contact, and the other in the direction of the tangent: now the mass of the body being denoted by m, the whole motive force of gravity and these three forces of reaction may be expressed by

$$mg,\ mf,\ mf',\ mf''.$$

6. Reducing the forces mf', mf'' to the directions of x and y, and tending to diminish dx and dy we have, by the principle determining the motions of the first class;

$$(1),\quad \frac{ddx}{dt^2}=-f'\sin\psi-f''\cos\psi,$$

$$(2),\quad \frac{ddy}{dt^2}=-f'\cos\psi+f''\sin\psi,$$

$$(3).\quad \frac{ddz}{dt^2}=f-g.$$

7. Calculating the moments of rotation about each of the axes of p, q, r, separately arising from the forces mf, mf', mf'', and adding the moments resulting from inertia, we have the following equation for the second class of motions;

$$(4),\quad C\frac{dp}{dt}=(A-B)qr+m\rho f''$$

$$(5),\quad A\frac{dq}{dt}=(B-C)pr+m\rho(f\cos\theta-f'\sin\theta)\cos\varphi,$$

$$(6).\quad B\frac{dr}{dt}=(C-A)pq-m\rho(f\cos\theta-f'\sin\theta)\sin\varphi.$$

8. Equations (1,) (2), (3) give the values of f, f', f'', as follows,

$$f=\frac{ddz}{dt^2}+g,$$

$$-f'=\frac{ddx}{dt^2}\sin\psi+\frac{ddy}{dt^2}\cos\psi,$$

$$-f''=\frac{ddx}{dt^2}\cos\psi-\frac{ddy}{dt^2}\sin\psi.$$

9. Put $\mu = C\frac{dp}{dt}+(B-A)\,qr$, $\mu'=A\frac{dq}{dt}+(C-B)\;pr$,

$$\mu''=B\frac{dr}{dt}+\;(A-C)\,pq:$$

and multiplping each of the equations (5), (6), by sin φ, cos ϕ separately, we have by addition and subtraction, the three equations (4), (5), (6) reduced into

$$\mu = m.\rho f''$$

$$\mu'\sin\varphi+\mu''.\cos\varphi = 0.$$

$$\mu'\cos\phi-\mu''\sin\; = m\rho\,(f\cos\theta-f'\sin\theta)\;;$$

in which, substituting the values of f, f', f'', as found in art. 8, we have,

(7), $$\mu+m\rho\left(\frac{ddx}{dt^2}\cos\psi-\frac{ddy}{dt^2}\sin\psi\right) = 0,$$

(8), $$\mu'\sin\varphi+\mu''\cos\varphi = 0,$$

(9), $$\mu''\sin\varphi-\mu'\cos\varphi+m\rho\left\{\left(\frac{ddx}{dt^2}\sin\psi-\frac{ddy}{dt^2}\cos\psi\right)\sin\theta+\left(\frac{ddz}{dt^2}+g\right)\cos\theta\right\}=0.$$

10. These three equations are general whatever be the forces of reaction mf, mf', mf'', and become applicable to any particular case by means of the equations of condition. In the present casé, the equations of condition of contact, $x=\rho\sin\theta$, and the two equations expressing the momentary rest of the element dm in contact with the horizontal plane. Computing these horizontal velocities by means of $dx\;dy$, pdt, qdt, rdt, we easily find including the differential of the equation of contact, the three equations of condition,

$$dx = \quad \rho\,pdt\cos\psi+\rho d\theta\sin\theta\sin\psi,$$

$$dy = -\rho\,pdt\sin\psi+\rho d\theta\sin\theta\cos\psi,$$

$$dz = \qquad\qquad \rho d\theta\cos\theta\;;$$

from which the square of the velocity of the centre of gravity is readily obtained in terms of p and $\frac{d\theta}{dt}$; for by squarng those equations and addition, we have

$$\frac{dx^2+dy^2+dz^2}{dt^2}=\rho^2\left(p^2+\frac{d\theta^2}{dt^2}\right).$$

11. We have now only to substitute the values of dx, dy, dz, in the equations, (7), (8), (9), and by an easy reduction these equations become,

(10), $$(\rho^2m+2A)\cdot\frac{dp}{dt}+\rho^2m.\ \frac{d\psi}{dt}\cdot\frac{d\theta}{dt}\sin\theta=0$$

(11), $$d.\left(\frac{d\psi}{dt}\sin^2\theta\right)+2pd\theta\sin\theta=0,$$

(12). $$(\rho^2m+A)\frac{dd\theta}{dt^2}-p(\rho^2m+2A)\frac{d\psi}{dt}\sin\theta-A\frac{d\psi^2}{dt^2}\sin\theta$$
$$\cos\theta+\rho mg\cos\theta=0.$$

Of those three equations which define the motion of the plate, the last is the same with that given by Mr. Nulty in his Prize solution; but the preceding two, the equations (10) and (11), are at variance with those given by that very ingenious mathematician. By the equation (10), p is generally variable, in his solution, p (or his r) is always constant.

12. When the plate is very nearly vertical, and $\frac{d\psi}{dt}$ extremely small, equ. 10, by omitting the 2d, term as an indefinitely small quantity of the second order, gives $p=$ constant; and therefore from equ. (11), we have

$$\frac{d\psi}{dt}-2p\cos\theta=k=\text{a const. quantity.}$$

In this case, if we put $n^2=\frac{12}{5}p^2-\frac{4}{5}\frac{g}{\rho}$, $n'=pk(\rho^2m+2A)$, and $u=\cos\theta$, eqa. 12 becomes

$$\frac{ddu}{dt^2}+n^2u+n'=0,$$

which determines the nature and extent of the oscillations made by the plate in this particular case.

13. When the values of p, θ, ψ are known in functions of the time t, the values of the co-ordinates x, y, o

centre of the plate are known by the equations in art. 9, which give the values of dx and dy. And when x and y are known, the co-ordinates $x, y,$ of the curve described on the horizontal plane by the point of contact are determined by the equations

$$x_{,} = x + \rho \cos\theta \sin\psi, \quad y_{,} = y + \rho \cos\theta \cos\psi.$$

Also if s be the length of the horizontal curve of contact, the value of s will be given by the equation $s = \rho\,\varphi$, in which s and φ are supposed to begin at the same instant.

14. When the circle rolls uniformly in the circumference of a circle, its angular velocity is $\frac{d\psi}{dt}$, which being denoted by V, and the radius of the circle described by R, we easily deduce from equ. (12) the following expression for determining V:

$$V^2 = \frac{4g \cot\theta}{6R - 5\rho\cos\theta},$$

which determines the angular and absolute velocities of the rolling circle about the centre of the circle described on the horizontal plane when the radius of the circle described and the inclination of the rolling circle are both given.

It is proper to remark, that in all the preceding calculations relative to the rolling circle, we have $A = B = \frac{1}{4}\rho^2 m$, and $C = 2A = \frac{1}{2}\rho^2 m$.

Professor Strong's first Solution to the Prize Question in No. VIII.*

Let the element dm of the plate be referred to three rectangular axes by the co-ordinates X, Y, Z; the first two in the horizontal plane on which the plate moves, and the third vertical to it, directed upwards. Put $g = 32.2$ feet, dt = const. differential of the time; then the general formula of Dynamics becomes in this

$$Sdm\left(\frac{d^2X\delta X + d^2Y\delta Y + d^2Z\delta Z}{dt^2} + g\delta Z\right) = 0 \ (1). \quad \text{Assume}$$

* This ought to have been published before Professor Strong's Solution in page 48.

$X = x + x'$; $Y = y + y'$; $Z = z + z'$; x, y, z, the co-ordinates of the centre of gravity of the plate referred to the axes before mentioned, and having the same origin, and x', y', z' respectively parallel to the former; but having the centre of gravity of the plate for their origin. I also put with La Place, Mec. Cel. page 73d, art. 26th, vol. 1st, $x' = ax'' + by''$; $y' = a'x'' + b'y''$; $z' = a''x'' + b''y''$; x'', y'' being referred to two diameters of the plate, which are at right angles to each other, their origin being at the centre of gravity, the terms involving the thickness of the plate in the expressions of x', y', z' have been neglected on account of their minuteness, and it should be noted that a, b, &c. are put respectively for the coefficients of x'', y'', &c. in the expressions for x', y', &c. as given at the place cited. Hence, I derive $Sdm\,\frac{dX^2+dY^2+dZ^2}{dt^2} = \text{M}\,\frac{dx^2+dy^2+z^2}{dt^2} + \frac{\text{MR}^2}{4} \times \frac{da^2+da'^2+da''^2+db^2+db'^2+db''}{dt^2}$; (in which R $=$ the radius of the plate and M $=$ the matter in the plate.)

Now by putting $d\varphi - d\psi \cos\theta = pdt$; $d\psi \sin\theta \sin\varphi - d\theta \cos\varphi = qdt$; $d\psi \sin\theta \cos\varphi + d\theta \sin\varphi = rdt$. I find $\frac{da}{dt} = bp - cr$; $\frac{da'}{dt} = b'p - c'r$ . . $\frac{da''}{dt} = b''p - c''r$; ∵ $\frac{db}{dt} = cq - ap$; $\frac{db'}{dt} = c'q - a'p$; $\frac{db''}{dt} = c''q - a''p$; hence $\frac{da^2+da'^2+,\ \&c.}{dt^2} = 2p^2+q^2+r^2$. Again, because the plate does not slide in any direction, but rolls on its line of contact on the horizontal plane with the velocity $\text{R}.\frac{d\varphi - d\psi\cos\theta}{dt}$, it is manifest that the centre of the plate is carried forward with the same velocity; and at the same time it turns about the line of contact with the velocity $\text{R}d\theta$, and these velocities are at right angles to each other ∵ $\frac{dx^2+dy^2+dz^2}{dt^2} = \text{R}^2 \times \left(\frac{(d\varphi - d\psi\cos\theta)^2 + d\theta^2}{dt^2}\right) =$ the square of the velocity of the centre of gravity of the plate. Hence by substitution

I have $Sdm\,\frac{dX^2+dY^2+dZ^2}{dt^2}=\text{M}.\frac{dx^2+,\ \&c.}{dt^2}+\frac{\text{MR}^2}{4}\times$ $\frac{da^2+,\ \&c.}{dt^2}=\frac{\text{MR}^2}{4}\times\left(\frac{6(d\varphi-d\psi\cos\theta)^2+d\psi^2\sin^2\theta+5d\theta^2}{dt^2}\right)$ (a). Also $Sdmg\delta Z=\text{MR}g\cos\theta\delta\theta$. Hence by the method of La Grange, Mec. Anal. vol. 1st, 2nd part, sec. 4th, the formula of Dynamics is transformed into one involving $\delta\varphi$, $\delta\psi$, $\delta\theta$, as independent variations whose coefficients put $=0$, by some very obvious reductions give $d.\left(\frac{d\varphi-d\psi\cos\theta}{dt}\right)=dp=0\ \because\ \frac{d\varphi-d\varphi\cos\theta}{dt}=p=$ const. (2). Also $d.\left(\frac{d\psi}{dt}\sin^2\theta-6p\cos\theta\right)=0\ \because\frac{d\psi}{dt}\sin^2\theta$ $-6\,p\cos\theta=$const.$=$c. At the origin of the motion $\frac{d\psi}{dt}=0$ $\because$ c$=-6\,p\cos\theta'$ ($\theta'=$the value of θ at the origin) $\because\frac{d\psi}{dt}$ $=6p\frac{(\cos\theta-\cos\theta')}{\sin^2\theta}$ (3). Also $\frac{5}{4}\frac{d^2\theta}{dt^2}-\frac{d\psi}{dt}\times\frac{\sin\theta}{4}$, $\left(\frac{d\psi}{dt}\cos\theta+6p\right)+\frac{g}{\text{R}}\cos\theta=0$, (4). This equation, by substituting the value of $\frac{d\psi}{dt}$ becomes $\frac{5}{4}\frac{d^2\theta}{dt^2}+9p^2\times$ $\left(\frac{\cos\theta'(1+\cos^2\theta)}{\sin^3\theta}-\frac{\cos\theta(1+\cos^2\theta')}{\sin^3\theta}\right)+\frac{g}{\text{R}}\cos\theta=0$, multiply by $d\theta$, integrate and correct by considering that at the origin of the motion $\frac{d\theta}{dt}=0$, and that $\theta=\theta'$ and there results $\frac{5}{8}\frac{d\theta^2}{dt^2}+\frac{9p^2}{2}\times\left(\frac{\cos\theta-\cos\theta'}{\sin\theta}\right)^2+\frac{g}{\text{R}}\times(\sin\theta$ $-\sin\theta')=0$, (5). From (5), I derive

$$dt=\frac{d\theta\cos\theta\sqrt{\ (5)}}{2\sqrt{\left(\frac{2g}{\text{R}}\cos^2\theta(\cos\theta'-\cos\theta)-9p^2(\sin\theta-\sin\theta')^2\right)}}$$ in

which θ, θ', are supposed to be reckoned from the vertical. Having found dt, in terms of, θ, and given quantities $d\psi$ and $d\varphi$, may also be found in terms of θ, and given quan-

tities by the method of series, equidistant ordinates, or by the aid of the conic sections. But I shall proceed in a less general manner to consider the case proposed in the question when the value of θ' is indefinitely small. If $(3p)^2 >$ $\frac{g}{R}$, and if $\frac{(3p)^2+\frac{g}{R}}{(3p)^2-\frac{g}{R}}$ is a finite quantity, then shall θ be a very small quantity. For, put $\frac{g}{R} = n$ and $(3p)^2 = p'^2$, and suppose that θ is so small that for $\cos^2 \theta$; $\cos \theta' - \cos \theta$, $\sin \theta - \sin \theta'$ under the radical in the denominator, we may put 1. $\frac{\theta^2 - \theta'^2}{2}$, $\theta - \theta'$, and for $\cos \theta$ in the numerator, we put 1. then $dt = \frac{(d\theta\sqrt{5})}{(2\sqrt{\ })n(\theta^2-\theta'^2)-p'^2(\theta-\theta')^2}$; this integrated and corrected gives $\theta = \left(\frac{p'^2-n\cos\left(2\sqrt{\frac{(p'^2-n)}{5}}\times t\right)}{p'^2-n}\right)$ $\times\theta'$, which gives $\theta = \theta'$ when $2\sqrt{\left(\frac{p'^2-n}{5}\right)}\,.\,t = 2mP$ and $\theta = \frac{p'^2+n}{p'^2-n}\times\theta'$ when $2\sqrt{\left(\frac{p'^2-n}{5}\right)}\,t = (2m+1)P$, (in which $m = 0$ or any positive integer number, and $P =$ the semi-circumference rad. (1); these show that θ is always very small when $\frac{p'^2+n}{p'^2-n}$ is a finite quantity, and θ' an infinitely small angle. The limits of θ being θ' and $\frac{p'^2+n}{p'^2-n}\times\theta'$ which are both very small on the suppositions here made. Again, $\frac{d\psi}{dt} = 2p'(\theta-\theta')$ $\therefore$ $d\psi = 2p'dt\,(\theta-\theta') = \frac{2p'n\theta'dt}{p'^2-n}$ $\left(1-\cos\left(2\sqrt{\frac{(p'^2-n)}{5}}\right)t\right)$; hence by integration and correction $\psi = \frac{np'\theta'\sqrt{(5)}}{(p'^2-n)^{\frac{3}{2}}}\times 2\sqrt{\left(\frac{p'^2-n}{5}\right)}t - \sin 2\sqrt{\left(\frac{p'^2-nt}{5}\right)}t$;

hence, p'^2-n, and t, being finite it is manifest that ψ will be very small; and hence may be put for its sine, and its cosine may be put $=1$. Suppose now that s denotes the portion of the curve described by the point of contact of the plate on the horizontal plane from the origin of the motion in the time t, and that the plate commenced its motion from the origin of the co-ordinates in the direction of the axis of X. Then $ds = \text{R}\,pdt$ nearly $= dX'$ very nearly, and $\because$ $X' = \text{R}pt$ nearly. Also $dY' = \text{R}pdt \sin\psi =$

$$\frac{\text{R}np'^2\theta'\sqrt{(5)},\,dt}{3\,(p'^2-n)^{\frac{3}{2}}}\times\left(2\sqrt{\left(\frac{p'^2-n}{5}\right)}t-\sin 2\sqrt{\left(\frac{5}{p'^2-n}\right)}t\right)\therefore$$

$$Y'=\frac{5\text{R}np'^2\theta'}{6(p'^2-n)^2}\times\left(2\times\frac{p'^2-n}{5}t^2-\left(1-\cos 2\sqrt{\left(\frac{p'^2-n}{5}\right)}\times t\right)\right)$$

. . X', Y' being the co-ordinates of the point of contact of the plate with the horizontal plane at the end of the time t. Also the co-ordinates of the centre of gravity are $x = X'$, $z = R$, $y = Y'+R\theta$ very nearly.

Whence the position of the plate on the suppositions made, becomes fully known at any time t, from the origin of the motion.

ARTICLE XXI.

SOLUTIONS

TO THE QUESTIONS PROPOSED IN ARTICLE XIX. NO. IX.

QUESTION I. (131.)—*By Mr. S. Hammond, Brooklyn.*

Given $\left\{\begin{array}{l} x^4-y^4=1280 \\ x^3y+xy^3=480 \end{array}\right\}$ to determine the values of x and y.

FIRST SOLUTION.—*By Mr. William Vogdes.*

Put $x = ny$; then the equations become

$$n^4y^4-y^4=1280, \text{ and } n^3y^4+ny^4=480;$$

$$\therefore y^4=\frac{1280}{n^4-1}, \text{ and } y^4=\frac{480}{n^3+n}.$$

Hence, by equality, we have

$$\frac{1280}{n^4-1}=\frac{480}{n^3+n};$$

whence, $6n^4-16n^3-16n=6$; from which we find $n=3$; consequently $x=ny=3y$. Substistute this in the first equation, and we have $81y^4-y^4=1280$; $\therefore y=2$, and consequently $x=3y=6$.

SECOND SOLUTION.—*By Mr. Gerardus B. Docharty.*

From the first of these equations, we have $x^2+y^2=\frac{1280}{x^2-y^2}$, and from the second $x^2+y^2=\frac{480}{xy}$; $\therefore \frac{480}{xy}=\frac{1280}{x^2-y^2}$, and by reduction, we have $x^2-\frac{8}{3}xy=y^2$. Hence, by completing the square, &c. we find $x=3y$; $\therefore$ we can readily obtain the values of x and y.

QUESTION II. (155.)—*By Mr. John Swinburne.*

Given $\left\{\begin{array}{l} y^6-x^4=23040000 \\ x^2y^3-2000x^2=4800y^3 \end{array}\right\}$ to determine the values of x and y by a quadratic.

FIRST SOLUTION.—*By Mr. James Docharty.*

Because 23040000 is the square of 4800, put $4800=a$, $2000=b$, then $y^6-x^4=a^2$, or $y^6=x^4+a^2$. Again, $x^2y^3-bx^2=ay^3$, or $x^2y^3-ay^3=bx^2$ by the Quest. whence $y^3=\frac{bx^2}{x^2-a}$, or $y^6=\left(\frac{bx^2}{x^2-a}\right)^2=\frac{b^2x^4}{x^4-2ax^2+a^2}$ $\therefore x^4+a^2=\frac{b^2x^4}{x^4-2ax^2+a^2}$; this cleared of fractions gives x^8-2ax^6+

$2a^2x^4 - 2a^3x^2 + a^4 = b^2x^4$, and by adding a^2x^4 to both sides, we have $x^8 - 2ax^6 + 3a^2x^4 - 2a^3x^2 + a^4 = b^2x^4 + a^2x^4 = (b^2+a^2)x^4 = c^2x^4$, where both sides are complete squares; and by extracting the roots we have, $x^4 - ax^2 + a^2 = cx^2$.

Again by subtracting ax^2 from both sides we have $x^4 - 2ax^2 + a^2 = cx^2 - ax^2 = (c-a)x^2 = d^2x^2$, when both sides are again complete squares. $\therefore$ $x^2 - a = dx$; hence by completing the square and extracting the roots, we find $x = 80$, and $y = (x^4 + a^2)^{\frac{1}{6}} = 20$.

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SECOND SOLUTION.—*By Mr. John M. Wilt.*

The equations may be thus expressed $y^6 - (4800)^2 = x^4$, and $\dfrac{4800y^3}{y^3 - 2000} = x^2$; the latter squared we have $\dfrac{4800y^6}{y^6 - 4000y^3 + (2000)^2} = y^6 - (4800)^2$. Put $a = 4800$, $c = 2000$, and $v = y^3$. By multiplication and transposition, we get $v^4 - 2cv^3 + (c^2 - 2a^2)v^2 + 2a^2cv = a^2c^2$, or $(v^2 - cv)^2 - 2a^2(v^2 - cv) = a^2c^2$. Completing the square, &c. $v^2 - cv = 48000000$, and $v = 8000$; whence $y = 20$ and $x = 80$.

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QUESTION III. (156)—*By Mr. John Swinburne.*

Given $\begin{cases} x^6y^3 - x^{12} = 9728 \\ x^2y^5 + x^8y^2 = 40320 \end{cases}$ to determine the values of x and y.

—

FIRST SOLUTION.—*By Mr. Patrick Lee, Brooklyn.*

Put $a = 9728$ and $b = 40320$.

From the first equation we get $y^3 = \dfrac{a + x^{12}}{x^6}$; then, $y^6 = \left(\dfrac{a + x^{12}}{x^6}\right)^2 = \dfrac{a^2 + 2ax^{12} + x^{24}}{x^{12}}$, divide the second equation into factors, viz. $x^2y^2(y^3 + x^6) = b$, and substitute for y^3 its

equal $\frac{a+x^{12}}{x^6}$, and we get $x^2y^2\times\left(\frac{a+x^{12}}{x^6}+x^6\right)=x^2y^2\times$ $\frac{a+x^{12}+x^{12}}{x^6}=b$, free from fractions; we then have $x^2y^2\times(a+x^{12}+x^{12})=bx^6$; divide both sides by x^2, then $y^2(a+2x^{12})=bx^4$; hence $y^2=\frac{bx^4}{a+2x^{12}}$, and $y^6=\left\{\frac{bx^4}{a+2x^{12}}\right\}^3$; therefore putting the two values of y^6 thus found to equal one another, we get $\frac{b^3x^{12}}{(a+2x^{12})^3}=\frac{a+x^{12}}{x^{12}}$. Multiply across, we get $(a+2x^{12})^3\times(a+x^{12})=b^3x^{24}$, which when actually multiplied, we have $a^5+8a^4x^{12}+25a^3x^{24}+38a^2x^{36}+28ax^{48}+8x^{60}=b^3x^{24}$, and by transposition $8x^{60}+28ax^{48}+38a^2x^{36}+\frac{25a^3-b^3}{8}\times x^{24}+8a^4x^{12}=-\frac{a^5}{8}$. Assume $z=x^{12}$, and restore the value of the coefficient of each term, we get $z^5+34048z^4+449511424z^3-5316666982400z^2+8955590927712256z=-1088999856809 8103296$; this solved gives $z=(x^{12})=4096$, hence $x=(4096)^{\frac{1}{12}}=2$; $y=\left\{\frac{a+x^{12}}{x^6}\right\}^{\frac{1}{3}}=216^{\frac{1}{3}}=6$. Therefore 2 and 6 are the values of x and y respectively.

Second Solution.—*By Mr. S. Hammond, Brooklyn.*

Put $x^2=vy$ and the given equations become $v^3y^6-v^6y^6=9728$, $vy^6+v^4y^6=40320$. From the first $vy^6=\frac{9728}{v^2-v^5}$. From the second, $vy^6=\frac{40320}{1+v^3}$. Whence $40320(v^2-v^5)=9728+9728v^3$. Dividing by 40320, $v^2-v^5=\frac{76}{315}+\frac{76}{315}v^3$, and by transposition, $v^5+\frac{76}{315}v^3-v^2=-\frac{76}{315}$. Whence $v=\frac{2}{3}$. Substituting this in the second equation above, we have $\frac{2}{3}y^6+\frac{16}{81}y^6=40320$. Therefore $y^6=$

$40320 \div \frac{70}{81} = 46656$ $\therefore y = 6$ and $x^2 = vy = \frac{2}{3} y = \frac{2}{3} \times 6 = 4$ and $x = 2$.

Question IV. (157.)—*By Mr. Michael Floy, Jun.*

It is required to find two numbers, such that their difference and the difference of their cubes, shall be rational squares.

First Solution.—*By Mr. John B. Newman.*

Let x and y represent the numbers; then, $x - y = \square$ (A) and $x^3 - y^3 = \square$ (B),

$$x^2 + xy + y^2 = \square \text{ (B} \div \text{A), (C).}$$

Let $x - 3y$ be the root, then $x^2 - 6xy + 9y^2 = x^2 + xy + y^2$;

$$\therefore 8y^2 = 7xy, \text{ or } 8y = 7x.$$

Hence, we see that the numbers to answer the conditions of equation (C), may be in the ratio of 8 to 7, and to answer the first, their difference must be a square; consequently, since 8 and 7 answer both, $8x^2$ and $7x^2$ will answer both conditions, where x may be taken any positive number whatever.

Second Solution.—*By Mr. A. H. Smith, Cazenovia.*

Let $x + n$ and $x - n$ be the numbers; then by cubing each and taking the difference of the cubes, we get $6nx^2 + 2n^3$. Now, if n be taken such that $2n$ is a square, and the above expression then be made a rational square by the usual methods, a variety of numbers may be found answering the conditions of the question; viz. $\frac{10}{9}$ and $\frac{6}{9}$, 10 and 6, 40 and 24, 288 and 252, &c.

Question V. (158.)—*By Mr. Solomon Wright.*

Required the height of a solid of a pyramidal form, the three sides of the base being 13, 14, and 15 respectively; and the three angles of the sides taken at the vertex, are found to be 30°, 40°, and 50°.

FIRST SOLUTION.—*By Mr. Eugenius Nulty, Phil.*

The slant sides of the pyramid may be determined from the sides of the base, and their opposite angles at the vertex. The solid content and area will then become known, and therefore the required perpendicular. See *Legendre's Geometry.** Note 5.

SECOND SOLUTION.—*By Mr. James Macully, N. Y.*

Having given the sides of the base, and their opposite angles, we can easily find the sides of the pyramid which include those angles. Now, let ABC be the base, and V the vertex, (the reader can easily supply the figure), then by plane Trigonometry, calculate the three angles BAC, BAV, and CAV of the triangle BAV, and also the perpendicular from V. Then it is plain, that if A be the centre of a sphere, the three angles above-mentioned will be the sides of a triangle on that sphere, and the angles of that triangle, will be the inclinations of the planes which contain the solid angle A. Find ∴ by spherical trigonometry the inclination of the planes ABC and ABV. Then as radius : the sine of the angle thus found :: the perpendicular of the triangle ABV : the required altitude.

Or thus. Let the three sides of the base be denoted by m, n, and P, and the sides of the pyramid by g, h, and f, and the required altitude by x. Then (Carnot on the Five Points, page 8th), we have

$$x^2(2m^2n^2+2m^2P^2+2n^2P^2-m^4-n^4-P^4)+f^4m^2+m^4f^2+g^4n^2$$
$$+n^4g^2+h^4P^2+P^4h^2$$
$$+m^2n^2P^2+m^2g^2h^2+n^2f^2h^2+P^2f^2g^2-m^2n^2f^2-m^2n^2g^2-m^2f^2g^2$$
$$-m^2f^2h^2$$
$$-m^2p^2f^2-m^2P^2h^2-n^2f^2g^2-n^2g^2h^2-n^2P^2g^2-n^2P^2h^2-P^2g^2h^2$$
$$-P^2f^2h^2=0.$$

THIRD SOLUTION.—*By Mr. Benjamin Wiggins.*

Let x, y, and z = the edges of the pyramid; then, by the properties of triangles

* Brewster's translation of this work for the use of the Military Academy West Point, is just published.

$$(13)^2 = x^2 - 2\cos 30^\circ xy + y^2,$$
$$(14)^2 = y^2 - 2\cos 40^\circ yz + z^2,$$
$$(15)^2 = z^2 - 2\cos 50^\circ zx + x^2,$$

from which the values of x, y, and z are found to be 10.82, 21.19, and 19.45, respectively. Whence the height is found to be 8.52.

QUESTION VI. (159.)—*By Mr. Farrand N. Benedict.*

Given $x^n(x^n+y^n+z^n+v^n) = a$, $y^n(x^n+y^n+z^n+v^n)=b$, $z^n(x^n+y^n+z^n+v^n) = c$, and $v^n(x^n+y^n+z^n+v^n) = d$, to determine the values of x, y, z, v.

FIRST SOLUTION.—*By Mr. John Swinburne, Brooklyn.*

The four given equations added together and the square root of the sum extracted, gives $x^n+y^n+z^n+v^n = \sqrt{(a+b+c+d)} = m$: this equation being divided into the four given equations, and the nth root of the result being taken, we obtain, respectively

$$x = \left(\frac{a}{m}\right)^{\frac{1}{n}},\ y = \left(\frac{b}{m}\right)^{\frac{1}{n}},\ z = \left(\frac{c}{m}\right)^{\frac{1}{n}},\ \text{and } v = \left(\frac{d}{m}\right)^{\frac{1}{n}}.$$

SECOND SOLUTION.—*By Mr. John Sullivan, U. S. Navy-Yard, Pensacola.*

Dividing the first three equations respectively by the last, we shall have

$$\frac{x^n}{v^n} = \frac{a}{d},\ \frac{y^n}{v^n} = \frac{b}{d},\ \text{and } \frac{z^n}{v^n} = \frac{c}{d};\ \text{then}$$

$$x^n = \frac{a}{d}v^n,\ y^n = \frac{b}{d}v^n,\ \text{and } z^n = \frac{c}{d}v^n.$$

Substituting these values in the first equation, and reducing, we shall find

$$(a+b+c+d)v^{2n} = d^2;\ \therefore v = \left\{\frac{d^2}{a+b+c+d}\right\}^{\frac{1}{2n}};$$

Hence

$$x=\left\{\frac{a^2}{a+b+c+d}\right\}^{\frac{1}{2n}},\ y=\left\{\frac{b^2}{a+b+c+d}\right\}^{\frac{1}{2n}},$$

$$z=\left\{\frac{c^2}{a+b+c+d}\right\}^{\frac{1}{2n}}.$$

QUESTION VII. (160.)—*By Mr. Farrand N. Benedict.*

Make $16x^5+8x^4+8x^3+5x^2+2x+1$, a rational square.

FIRST SOLUTION.—*By Mr. Sears C. Walker.*

Assume the side $= 1+x+2x^2+2x^3$; then

$$16x^5=4x^6+8x^5;\ \therefore 4x^6=8x^5,$$

and consequently $x=2$.

The solutions of almost all the Contributors were similar to this.

QUESTION VIII. (161.)—*By Mr. Jacob Borton.*

A circle and its diameter being given in position and magnitude, it is required to draw a tangent to the circle at one end of the diameter and a line from the other end of the diameter; such that the part of the line intercepted by the circle shall be equal to the diameter.

FIRST SOLUTION.—*By Mr. E. Lynch, N. Y.*

Let ACB be the given circle* and AB its diameter. From the point A draw the line AC, terminating in the circumference equal to the tangent AF, and from the point B at the other end of the diameter, draw the line BE parallel to AC, and the thing is done. For the chords AC, BE being parallel, and at equal distances from the centre, are equal; therefore $BE=AC=AF$.

* The figure can be easily supplied by the reader.

Cor. 1. In the same manner it may be shown that the tangent BD would be equal to a line drawn from the point A parallel to the chord BC.

Cor. 2. If AC and BE be produced to D and F, the triangles ABD, ABF, will be similar and equal; $\therefore$ $CD = EF$.

Second Solution.—*By Mr. J. Thomson.*

If a perpendicular be drawn from one extremity of the diameter upon the intercepted line, it will meet it at the point intercepted by the circle, and divide the whole into similar triangles, putting tang. $= x$, diam. $= a$, we have the following proportion, $x : a :: a : \frac{a^2}{x} =$ hypothenuse, whence $\frac{a^4}{x^2} = a^2 + x^2$, and $x = \sqrt{\left(\frac{a}{2}\sqrt{5} - \frac{a}{2}\right)}$.

Question IX. (162.)—*By Mr. James Sloane, N. J.*

Given two right lines in position forming an acute angle and a point adjacent to them; to describe a circle that shall have its centre in one of the lines, it shall pass through the given point, and to which the other line shall be a tangent.

First Solution.—*By Mr. M. O'Shannessy, N. Y.*

Let P be the given point, and AB, AC the two given lines making a given angle α; and let the angle $PAC = \varphi$. Put the given line $AP = c$, and $AC = x$; then let C be the centre of the circle, touching AB in B. Now, by trigonometry, $c^2 + x^2 - 2cx \cos\varphi = CP^2$, and its equal $CB^2 = \sin^2 \alpha x^2$. These values equated give $x^2 + 2ax = c^2$, $\left(a \text{ being put for } \frac{c \cos\varphi}{\sin^2 \alpha}\right)$, which can be readily constructed geometrically.

SECOND SOLUTION.—*By Doctor Adrain.*

Let a and b be the co-ordinates of the given point, $x =$ abscissa of the centre, $s =$ sine of the given angle; then it is evident that the required circle is determined by the quadratic

$$(a-x)^2+b^2 = s^2 x^2.$$

QUESTION X. (163.)—*By Mr. Thomas J. Megear.*

It is required to find an arc, such that the rectangle of its versed sine and the versed sine of its complement shall be a maximum.

FIRST SOLUTION.—*By Mr. Eugenius Nulty, Phil.*

Let ϕ be the arc; then $(1 \mp \cos\varphi) \cdot (1 \mp \sin\varphi) =$ max.; from which there results $\sin\varphi = \cos\varphi$, or $\sin\varphi + \cos\varphi = \pm 1$. We have therefore $\varphi = \frac{\pi}{4}$ or $\frac{5\pi}{4}$, which is the arc required.

SECOND SOLUTION.—*By Professor Strong.*

Let φ denote the arc sought radius equal, (1); then, (per quest.) $(1-\cos\varphi)(1-\sin\varphi) =$ max. For $1-\cos\varphi =$ versin. and $1-\sin\varphi =$ coversin ϕ; therefore $d\varphi \sin\varphi (1-\sin\varphi) - d\varphi \cos\varphi (1-\cos\varphi) = 0$; $\therefore (\sin\varphi - \cos\varphi) \times \{1-(\sin\varphi+\cos\varphi)\} = 0$. Of the factors in this equation, it is evident that $\sin\varphi - \cos\varphi = 0$ is the only one which applies. This gives $\sin\varphi = \cos\varphi$; $\therefore$ if our views are confined to the first quadrant the arc sought $=$ 45.°

Note. The product is a maximum in the first and third quadrant, and a minimum in the second and fourth; and so on for the successive quadrants. The odd quadrants giving maxima, and those that are denoted by even numbers giving minima: and it may be remarked that when the number which denotes the quadrant is of the form $4n+3$, that the product becomes an absolute maximum.

THIRD SOLUTION.—*By Mr. Silas Warner.*

Let radius equal unity and versed sine of the required arc $= x$; then, by the property of right angled triangles, the versed sine of its complement equal to $1-\sqrt{(2x-x^2)}$; $\therefore\ x\times\{1-\sqrt{(2x-x^2)}\} =$ max. Hence, by differentiation and reduction, we find $2x^3-6x^2+5x-1=0$.

This divided by $x-1$, gives $2x^2-4x+1=0$, a quadratic, which, when solved, gives $x=1-\sqrt{\frac{1}{2}}$, the versed sine of $45°$.

FOURTH SOLUTION.—*By Doctor Bowditch.*

Let $x =$ the angle sought, then by the question $(1-\cos x).(1-\sin x) =$ max. The differential of this equated to zero and divided by dx, gives

$$(\cos x-\sin x).(-1+\cos+\sin x)=0.$$

The first factor gives, $x = 45°$ or $225°$, which correspond to a maximum. The other factor is satisfied by $x = 0$, or $x = 90°$, corresponding to a minimum,

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QUESTION XI. (164.)—*By Mr. James Divver.*

Given the base, the difference of the angles at the base, and the line that bisects the vertical angle; to determine the triangle.

FIRST SOLUTION.—*By Mr. William Lenhart, Penn.*

Let AB be the given base, (the figure can be easily supplied) which bisect by the perpendicular CD. Make the angle DCE equal to half the difference of the angles at the base, and make CE equal to the line bisecting the vertical angle. Through E draw ED, parallel to AB, cutting CD in D, and CD will evidently be equal to the perpendicular of the required triangle. Hence the problem may be constructed as Prob. 15. Appendix to Simpson's Algebra.

SECOND SOLUTION.—*By* Ομικρον, N. C.

Let a=semibase, b=length of the line bisecting the vertical angle, and $x=\frac{1}{2}$ vertical angle; then cos x : sin x :: a : a tan x = that part of the diameter of the circumscribing circle which falls perpendicularly below the base; and if A = diff. of angles at the base, cos A : sin A :: a tan x : a tan A. tan x = segment of the base contained between the said diameter, and the bisecting line; also cos A : 1 :: a tan x : $\frac{a \tan x}{\cos A}$ = that part of the bisecting line below the base limited by the diameter; hence the segments of the base are a (1 – tan A. tan x) and a(1 + tan A. tan x); consequently $a^2(1-\tan^2 A. \tan^2 x) = \frac{ab \tan x}{\cos A}$, a quadratic equation, from which the vertical angle is known and consequently the dimensions of the triangle.

QUESTION XII. (165.)—*Mr. William H. Sidell.*

The base, difference of the squares of the sides, and the sum of the tangents of the angles at the base being given, to construct the triangle.

FIRST SOLUTION.—*By Mr. Marcus Catlin.*

The difference of the squares of the sides evidently equals the difference of the squares of the segments of the base made by the perpendicular. Hence the segments of the base are known. Put r and r' for those segments, a= base, s = the sum of the tangents at the base, and x = the perpendicular. Then we have $\frac{x}{r}$= one of the tangents, and $\frac{x}{r'}$ = the other $\therefore \frac{x}{r}+\frac{x}{r'} = s$, by the question. Hence $r'x+rx = rr's \therefore x = \frac{rr's}{r+r'} = \frac{rr's}{a}$.

Hence the triangle is easily constructed.

SECOND SOLUTION.—*By Mr. P. J. Rodriguez, Gosport, Virg.*

Let the sides AB and BC be represented by m and n respectively, the base $AC = b$, one of the segments AD of the base, made by the perpendicular BD, equal to x, the sum of the tangents of the angles A and $C = s$, and d the difference of the squares of the sides. Then on account of the perpendicular BD, we have

$$m^2 - x^2 = n^2 - (b-x)^2 ; \therefore m^2 - n^2 = 2bx - b^2,$$

and consequently, $x = \dfrac{d+b^2}{2b}$.

We have also $x \tan A = (b-x) \tan C$.

Substituting in this equation the values of x and of $\tan A = s - \tan C$, we have

$$\frac{d+b^2}{2b}\left(s - \tan\ C\right) = \left(b - \frac{d+b^2}{2b}\right)\tan C ;$$

hence, $\tan\ C = s\left(\dfrac{d+b^2}{2b^2}\right)$.

QUESTION XIII.—*By Mr. James Macully, N. Y.*

Given the sum of the two sides containing the vertical angle of a plane triangle, and the radius of its inscribed circle to construct it; when that part of the line bisecting the vertical angle included between the base and the centre of the inscribed circle is a maximum.

FIRST SOLUTION.—*By Dr. Adrain.*

Let a equal the given sum of the sides, r = the given radius of the inscribed circle, θ = half the vertical angle, and m = the straight line which must be a maximum; then we have m in terms of θ by the equation

$$m = \frac{r}{a}\left\{\frac{a}{\sin\theta} - 2r\frac{\cos\theta}{\sin^2\theta}\right\}.$$

Now, putting $\frac{a}{2r} = n$, and equating to zero, the differential of the value of m, we have the quadratic

$$\tan^2 \theta - n \tan \theta + 2 = 0.$$

SECOND SOLUTION.—*By Mr. Eugenius Nulty.*

Let x be the base of the triangle; y the difference of the other sides; s the given sum of those sides, and r the given radius of the circle. By a known theorem we have $4r^2 = (x^2 - y^2) \cdot \frac{s-x}{s+x}$; and the segment of the base between the point of contact and the line bisecting the vertical angle is $2y \cdot \frac{s-x}{s}$, which is evidently a maximum. Wherefore $y^2 \cdot (s-x)^2$, or by eliminating y^2,

$$x^2(s-x)^2 - 4r^2(s^2-x^2) = \text{max.}$$

From this expression we find by differentiation $(2x-s) \cdot (s-x) = 4r^2$, the construction of which determines the triangle.

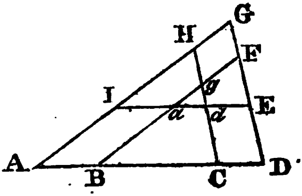
QUESTION XIV. (167.)—*By* Ομικρον, *N. C.*

A gentleman of North Carolina having purchased a quantity of land in the form of a triangle, and being unable to select a spot suitable for building, has directed a surveyor to lay out a triangular lot for this purpose, by running lines parallel to the several sides of the triangle in such a manner, that the whole together, with the two parts into which it is divided by the parallel lines, may successively constistitute an arithmetrical, geometrical, and an harmonical ratio. Required the ratio of the lines to their opposite sides, and also the dimensions of the triangular lot, those of the whole being known.

FIRST SOLUTION.—*By the Proposer.*

Let ADG be the given triangle, EI, FB, and HC the parallel lines dividing the triangles as required; then ADG : IEG :: AD^2 : IE^2; but by hypothesis IEG + ADG = 2IEDA = 2ADG — 2IEG, whence ADG = 3IEG, hence 3IEG :

$IEG :: AD^2 : IE^2 :: 3 : 1$. Again, $ACH : ADG :: CH^2 : DG^2$, and by hypothesis $ACH : ADG - ACH :: ADG - ACH : ADG$, and compos. $ACH : ADG :: ADG - ACH : 2ADG - ACH :: CH^2 : DG^2$; hence, and from the first analogy divid. $ADG - ACH : ADG :: CH^2 : DG^2 - CH^2 :: DG^2 - CH^2 : DG^2$; whence



$DG \times CH = DG^2 - CH^2$ or $DG^2 = (DG + CH) . CH$, $\therefore$ CH may be found by Prob. 6, Simpson's Select Exercises. Once more $FBD : GAD :: FB^2 : AG^2$, and by hypothesis, $FBD : GAD :: GAD - 2FBD : FBD :: FB^2 : AG^2$; hence, $GAD - 2FBD : 2FBD :: FB^2 : 2AG^2$, and $GAD : FBD : FB^2 + 2AG^2 : AG^2 :: AG^2 : FB^2$, $\therefore AG^4 = (FB^2 + 2AG^2) . FB^2$. To these equals add AG^4, and we have $2AG^4 = (AG^2 + FB^2)^2$, whence FB may be determined. Lastly, because IE is known, and the triangles IEG, ADG are similar, IG is known, and consequently $AI = aB$. Similarly $GH = gF$, $\therefore$ ag and the triangle adg, are known.

Cor. I. When the parts are in Arithmetical Progression, then $AB : IE :: 1 : \frac{1}{3}\sqrt{3}$.

II. When they are in Geometrical Progression, then $DG : CH :: 1 : \frac{1}{2}(\sqrt{5} - 1)$; but

III. When they are in Harmonical Progression, then $AG : BF :: 1 : (\sqrt{2} - 1)^{\frac{1}{2}}$.

Second Solution.—*By Mr. Eugenius Nulty:*

Let a be the area of the given triangle; ax, ay, az, the areas of the similar triangles cut off by the parallel lines, and $\therefore$ $a(1-x)$, $a(1-y)$, $a(1-z)$ the areas of the corresponding trapezoids. We shall then have $2 - x = 2x$, $1 - y = y^2$, and $2z - z^2 = 2 - z$; by virtue of which and the known relation between the areas and sides of similar triangles all the required parts may be easily determined.

THIRD SOLUTION. — *By Dr. Adrain.*

Let the three sides AG, GD, DA, (see the diagram to the first solution,) be denoted by a, b, c; the three segments AH, GE, DB by x, y, z; so that HC, EI, BF parallel to the given sides may divide the triangle in arithmetical, geometrical ratio, and form the boundaries of the triangular lot agd. The three areas AGD, AHC, and $HGDC$ being arithmeticals and proportionals to the three

$$a^2,\ x^2,\ a^2-x^2;$$

therefore $2x^2 = a^2+a^2-x^2$, or $3x^2 = 2a^2$; whence $x = a\sqrt{\frac{2}{3}} = \cdot 816a$.

Again, because AGD, GIE, and $AIED$, are geometrical proportionals: therefore, b^2, y^2, and b^2-y^2 are proportionals; whence $y^4+b^2y^2 = b^4$; $\therefore\ y = b\sqrt{\left\{\frac{\sqrt{5}-1}{2}\right\}}$ $= .786\,b$.

Thirdly, DAG, DBF, $AGFB$ are harmonical, and proportional to c^2, z^2, c^2-z^2; whence $4c^2z^2-z^4 = 2c^4$, and consequently,

$$z = c\sqrt{\{2-\sqrt{2}\}} = .766c.$$

From which values of x, y, z, the sides and area of agd are easily found.

In general whatever be the conditions by which x, y, z are determined by separate ratios to a, b, c, let $x = na$, $y = n'b$, $z = n''c$, and the three sides of the triangular lot will be

$$ag = (n+n'+n''-2)a,$$
$$bd = (n+n'+n''-2)b,$$
$$ad = (n+n'+n''-2)c.$$

and therefore the area of agd = the area of $AGD \times (n+n'+n''-2)^2$.

In the present case, $n = .816$, $n' = 786$, $n'' = .766$, hence $n+n'+n''-2 = .368$, and therefore the area $agd = .1354\ AGD$.

If $n = n' = n''$, then $agd = (3n-2)^2.\ AGD$.

If $n+n'+n'' = 2$, then the parallels to the sides intersect in one point.

QUESTION XV. (168.)—*By Henry Vose,* Esq.*

It is required to prove that, if n be a prime number, $v^n = x^n \pm y^n$ is or is not rationally impossible, which has never been proved satisfactorily.

PARTIAL SOLUTION.—*By J. Ingersoll Bowditch.*

If the equation is $v^n \pm y_n = x^n$, we may, besides considering x as prime, also suppose n to be prime, and x, y, n prime to each other, as has been proved by Barlow and others.

CASE 1. We have $v^n + y^n = x^n$ and $v + y = x^a$. But $v < x$ and $y < x$ $\therefore$ $v + y < 2x$. Hence $x^a < 2x$ $\therefore$ $x^{a-1} < 2$. But a must be a prime whole number and a must also be an integer number. Consequently $v^n + y^n = x^n$ *is impossible.*

CASE 2. Also it is required that $v^n - y^n = x^n$.

Then $v - y = x^c$ $\therefore$ $v = x^c + y$ $\therefore$ $(x^c + y)^n - y^n = x^n$ $\therefore$ $x^{cn} < x^n$.

But c must be a whole number, or nothing; it cannot be a fraction.—Hence $c = 0$ requires that

$$(y+1) - y^n = x^n;$$

x and n being prime numbers.

The equation $v^n = x^n \pm y^n$ is rationally impossible, when n is any integer greater than 2; which is proved in Chap. V. Book, I. of Barlow's Treatise on the Theory of Numbers. This theorem was first discovered by FERMAT, but he never published a demonstration of it; see the Appendix to *Euler's* Algebra, Vol. II. Note 11, page 471.

QUESTION XVI. (169).—*By Mr. Charles Farquhar.*

If on the ordinate PM produced, of any curve AM, the distance MM′ be taken always equal to the abscissa AP; if AM, and AM′ be joined, then will the area of the seg-

* In the last Number of the Diary this name was printed incorrectly.

ment cut off by AM from the given curve, be always equal to the segment cut off by AM′ from the curve, which is the locus of M′. Required the demonstration.

—

FIRST SOLUTION.—*By Mr. William H. Sidell, N. Y.*

Let x and y represent the co-ordinates of the curve AM′; then its area will be equal to the integral of ydx, and consequently that of the segment AM $=\int ydx-\frac{xy}{2}$. Also the area of the curve AM′ $=\int ydx+\frac{y^2}{2}$, and hence that of the segment AM′ $=\int ydx+\frac{y^2}{2}-(x+y)\frac{y}{2}=\int ydx-\frac{xy}{2}$, whence the truth of the proposition is manifest.

—

SECOND SOLUTION.—*By Mr. Gerardus B. Docharty.*

Let $x=Ap$, $y=PM$, then $x+y=PM'$, we have $\int ydx=$ area of the curve AMP and $\int(xdx+ydx)=$ area of $AM'P$. But $\frac{xy}{2}=$ area of the triangle APM and $\frac{x^2+xy}{2}=$ area of the triangle APM'. Therefore $\int ydx-\frac{xy}{2}=$ area of the segment cut off by AM and $\int(xdx+ydx)-\frac{x^2+xy}{2}=$ segment cut off by AM'. These by the question are said to be equal

$$\therefore \int ydx-\frac{xy}{2}=\int xdx+\int ydx-\frac{x^2+xy}{2}.$$

but by transposition $\frac{x^2+xy}{2}-\frac{xy}{2}=\int xdx$, or $x^2=\int 2xdx$;

$\therefore x^2=x^2$, by Integration;

which shews that the expressions are identical, the demonstration required.

COR. It is proper to observe that the segments are equal when the prolongation of the ordinate PM is any multiple of the abscissa AP.

QUESTION XVII. (170.)—*By S. of G.*

Given the base, the vertical angle, and the angle subtended by the base at a point which divides the perpendicular of the triangle in a given ratio. Required to construct the triangle geometrically.

—

FIRST SOLUTION.—*By Mr. Benjamin Pierce, Jun.*

Call the base b, the vertical angle v, the other given angle v', the perpendicular x, the lower part cut off ax, and we have, since $\text{tang}\,(v+v') = \frac{\text{tang}\,v + \text{tang}\,v'}{1 - \text{tang}\,v\ \text{tang}\,v'}$, $\frac{2bx}{b^2-y^2-x^2} = \text{tang}\,v$, $\frac{2bax}{b^2-y^2-a^2x^2} = \text{tang}\,v'$, $b^2-y^2 = x^2 + \frac{2bx}{\text{tang}\,v} = a^2x^2 - \frac{2bax}{\text{tang}\,v'}$, $x = \frac{2b}{1-a^2}\left(\frac{1}{\text{tang}\,v} - \frac{a}{\text{tang}\,v'}\right)$.

—

SECOND SOLUTION.—*By M. O'Shannessy.*

If upon the given base we describe segments of circles containing angles equal to the given ones, their radii and centres are given. Let then α, β, α', β', be the co-ordinates of those centres, and r and r' the corresponding radii. Let the origin of the co-ordinates be at the middle of the base, a perpendicular to which will pass through their centre, and will be the y, axis; then the equations of both centres being $(x-\alpha)^2+(y-\beta)^2=r^2$, and $(x-\alpha')^2+(y'-\beta')^2 = r'^2$. Since the same value x corresponds to the ordinates y and y', and because $y' = \frac{my}{n}$, this expression when divided by y (because $r'^2+\beta^2 = \beta'^2+r^2$) gives $y = \frac{m(m\beta'-n\beta)}{m^2-n^2}$, a simple equation which is readily constructed.

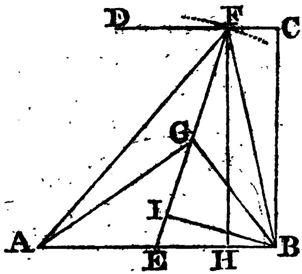
—

QUESTION XVIII. (171.)—*By Mr. William Lenhart.*

In a triangle whose base is 50, and perpendicular from the vertical angle on the base 48; it is required to find a point from which if two straight lines be drawn to the angles

at the base, the sum of the squares of the sides of the triangle may be to the sum of the squares of the two straight lines thus drawn, as 5 to 2.

First Solution.—*By the Proposer.*



Solution.—From the extremity of the given base AB, erect a perpendicular BC equal to the one given. Through C draw CD parallel to AB, and on bisecting AB in E, take it as a centre, and with the distance AB describe an arch cutting DC in F. Join FA, FE, FB, and bisect FE in G; so shall G be the point, and AFB the triangle required.

Demonstration.—Join GA, GB; and since AE = EG = EB, it follows that the triangle AGB is right angled. Again, FE is drawn to the middle of the base AB, whence $(AF)^2 + (FB)^2 = 2(AE)^2 + 2(FE)^2$. But $2(FE)^2 = 8(AE)^2$, then $(AF)^2 + (FB)^2 = 10(AE)^2$ and $(AG)^2 + (GB)^2 = (AB)^2 = 4(AE)^2$, consequently $(AF)^2 + (BF)^2 : (AG)^2 + (BG)^2 :: 10(AE)^2 : 4(AE)^2$, or as 5 : 2.—Q. E. D.

Calculation.—Draw the perpendiculars BI, FH. The triangles BEI, EFH being similar, we have $EF = AB = 50 : FH = BC = 48 :: EB = 25 : BI = 24$; and $\sqrt{((BE)^2 - (BI)^2)} = EI = 7$, then $GI = 18$, $GB = \sqrt{((BI)^2 + (GI)^2)} = 30$; and $AG = 40$. Again $\sqrt{((FE)^2 - (FH)^2)} = EH = 14$, then $AH = 39$, $HB = 11$, $AF = \sqrt{((AH)^2 + (HF)^2)} = 15\sqrt{(17)}$ and $BF = \sqrt{(\overline{HB}^2 + \overline{FH}^2)} = 5\sqrt{(97)}$.

Second Solution.—*By Dr. Adrain.*

Let a = the straight line drawn from the vertex to the middle of the base, b = half the base, and r = the distance of the middle of the base to the point sought. Determine r from the condition

$$5r^2 = 2a^2 - 3b^2,$$

and the circumference to the radius r and centre in the middle of the base will be the locus required.

Question XIX. (172.)—*By Mr. M. O'Shannessy, A. M.*

A spherical body of a given magnitude is projected in a given direction with a given impetus; required the equation of the curve that will constantly touch it in every point of its trajectory, both curves being in the same plane.

First Solution.—*By Dr. Adrain.*

Let ϕ = the angle which the normal of the curves makes with the axis of x, p = the parameter of the parabolic trajectary, then x and y being the co-ordinates, x being reckoned on the axis of the parabola from its vertex, we have the two equations of the curve as follows,

$$x = \frac{1}{4}p \tan^2\phi - a \cos\phi,$$

$$y = \frac{1}{2}p \tan\phi + a \sin\phi.$$

In which a, the radius of the sphere, must be taken positively for the curve on the outside of the parabola, and negatively for the curve within the parabola.

Eliminating ϕ from these equations, we have the equation of the curve in x and y: but if the tangent, area, &c. of the curve be required, the two equations in ϕ will be found much more useful than the single equation in x and y, which is of the sixth degree. If instead of x we put $x - 1$, so that the origin of x may be in the axis of the parabola produced to a distance equal to unity from the vertex, and denoting the parameter of the parabola by 4, the equation of the required curve, putting $b = 4 - a^2$, is found

$$(xy^2 - 4x^2 + bx + 9a^2)^2 = ((y^2 - 4x + b)^2 - 12a^2x).(4x^2 + 3y^2 - 12x + 3b).$$

Second Solution.—*By Mr. Eugenius Nulty, Phil.*

Let $b^2 = 2ma$ be the equation of the parabola described

by the centre of the spherical body, of which the radius is c; and let (x, y) be a point in the required curve. The equation of the great circle of the sphere passing through $(x, y,)$ and having its centre at the point (a, b) is

$$(2mx-b^2)^2+4m^2(y-b)^2=4c^2m^2;$$

and differentiating relatively to the ordinate b, the equation of c regarded as a normal to the parabola at (a, b), will be

$$b(2mx-b^2)+2m^2(y-b)=0.$$

From these equations we obtain, by assuming $m-x=r$, $x^2+y^2-c^2=s$, and eliminating b,

$$4r^4s-4mr^3y-8r^2s^2+36mrsy^2+4s^3-27m^2y^4=0;$$

the equation of the required curve, and which may be easily expressed in terms of x and y, by means of the assumed values of r and s.

QUESTION XX. (173.)—*By Professor Strong, Rutgers College.*

Determine the equation of the curve which is such that its length varies as its radius of curvature corresponding to its variable length.

FIRST SOLUTION.—*By Doctor Bowditch.*

The absciss being x, ordinate y, curve s, the radius of curvature supposing ds constant is $r=\frac{ds.dx}{ddy}$. This per question is $=ns+b$, n and b being constant.

Let z be the angle formed by the elements ds, dx, so that $dx=ds\,.\cos z$, $dy=ds\,.\sin z$, $ddy=dsdz\,.\cos z$. $\therefore ns+b=\frac{ds}{dz}$ or $\frac{ds}{ns+b}=dz$, whose integral is $ns+b=ce^{nz}$, c being constant. Its differential divided by n gives $ds=cdze^{nz}$; substitute this in dx and dy, and integrate them, gives

$$x=\frac{ce^{nz}}{nn+1}\Big(n\cos z+\sin z\Big);\ y=\frac{ce^{nz}}{nn+1}\Big(n.\sin z-\cos z\Big),$$

$$r=ce^{nz}$$

whence the curve may be constructed.

SECOND SOLUTION.—*By Doctor Adrain.*

Let z be the curve line, ρ its radius of curvature, x, y, its rectangular co-ordinates, $\phi =$ the angle contained by ρ and x; and n being any invariable quantity, let $\rho = nz$.

In general $d\phi = \frac{dz}{\rho}$, therefore $nd\phi = \frac{dz}{z}$, whence $z = c.e^{n\varphi}$, c being an arbitrary constant, and e the number of which the hyperbolic logarithm is unity.

But $dx = dz \sin\phi$, and $dy = dz \cos\phi$, that is,

$$dx = nce^{n\varphi}d\phi \sin\varphi,\ dy = nce^{n\varphi}d\phi \cos\phi,$$

the integrals of which are

$$x - \mathrm{c} = \frac{n}{2}ce^{n\varphi}(\sin\varphi - \cos\varphi),\ y - \mathrm{c}' = \frac{n}{2}ce^{n\varphi}(\sin\varphi + \cos\varphi).$$

From these two equations, since $z^2 = c^2e^{2n\varphi}$, we have

$$(x-\mathrm{c})^2 + (y-\mathrm{c}')^2 = \frac{n^2z^2}{2}.$$

Let $r =$ the radius vector from the point of which the co-ordinates are c, c′, and this equation becomes

$$r^2 = \frac{n^2z^2}{2},\ \text{or}\ nz = r\sqrt{2},$$

which shows that the curve z, and radius vector r, as well as their differentials, have a constant ratio, and therefore the required curve is the equiangular or logarithmic spiral.

QUESTION XXI. (174.)—*By the same.*

Let any number of parabolas whatever have the same vertex; it is required to find the nature of the curve which, cutting all the parabolas, shall cut off equal arcs, reckoned from the common vertex to the required curve.

FIRST SOLUTION.—*By Mr. J. Ingersoll Bowditch.*

Let x and y be the co-ordinates of the parabola.

$$\text{Then } yy = px \therefore p = \frac{yy}{x}.$$

We have for the length of the parabolic arc

$$\frac{1}{4}p\left\{\frac{2y}{p}\sqrt{\left(1+\frac{4y^2}{p^2}\right)}+\text{hyp.log.}\left(\frac{2y}{p}+\sqrt{\left(1+\frac{4y^2}{p^2}\right)}\right)\right\},$$

substitute for p its value found above, and we obtain

$$\frac{1}{4}\frac{y^2}{x}\left\{\frac{2x}{y}\sqrt{\left(1+\frac{4x^2}{y^2}\right)}+\text{hyp.log.}\left(\frac{2x}{y}+\sqrt{\left(1+\frac{4x^2}{y^2}\right)}\right)\right\}$$

$=$ constant, the required equation.

Second Solution.—*By Mr. Eugenius Nulty.*

Let the equation of the parabola be $y^2 = 2mx$, and the arc cut off a We shall then have for the length of any arc a terminating at the point $(x, y,)$

$$a = \frac{y}{2m}\sqrt{(m^2+y^2)}+\frac{m}{2}\log.\left\{\frac{y+\sqrt{(m^2+y^2)}}{m}\right\};$$

from which by eliminating the semiparameter m, the equation of the curve will be found

$$2a = \sqrt{(y^2+4x^2)}+\frac{y^2}{2x}\log.\left\{\frac{2x+\sqrt{(y^2+4x^2)}}{y}\right\}.$$

If we put $x = r\cos\phi$, and $y = r\sin\varphi$, the polar equation will be

$$r = 4a\left\{2a\sqrt{(1+3\cos^2\varphi)}+\frac{\sin\varphi}{\cos\varphi}\log.\right.$$
$$\left.\left(\frac{2\cos\varphi+\sqrt{(1+3\cos^2\varphi)}}{\sin\varphi}\right)\right\}^{-1}.$$

The curve corresponding to this equation meets the axes of x and y, perpendicularly at the distance a from the origin. Its figure resembles a semicircle, of which the diameter coincides with the axis of y.

Question XXII. (175)—*By Philotechnus.*

Two simple pendulums, the length of which are a and b inches, are suspended to the same point and have their oscillating points connected by an inflexible line or wire, equal

c inches; find the tension of this line when the compound pendulum moves by the action of gravity.

First Solution.—*By Mr. Eugenius Nulty.*

Let m and m' be the masses, supposed to be concentrated at the extremities of the given lines a and a' or b; and let φ and φ' be the angles formed by these lines and a vertical passing through the point of suspension; γ the angle opposite to the connecting line c; T the tension of this line, and g the force of gravity.

In the instant dt, the forces lost perpendicularly to the lines a, a' are $m\left(a\dfrac{d^2\varphi}{dt^2}-g\sin\varphi\right)$, $m'\left(a'\dfrac{d^2\varphi'}{dt^2}-g\sin\varphi'\right)$; and these forces and the tension T multiplied respectively by the elements of their directions $a\delta\varphi$, $a'\delta\varphi'$, $\dfrac{aa'}{c}\sin\gamma$ $(\delta\varphi'-\delta\varphi)$, give in consequence of the independence of the arbitrary variations $\delta\varphi$, $\delta\varphi'$,

$$m\left(a^2\frac{d^2\varphi}{dt^2}-ag\sin\varphi\right)-T\frac{aa'}{c}\sin\gamma=0,$$

$$m'\left(a'^2\frac{d^2\varphi'}{dt^2}-a'g\sin\varphi'\right)-T\frac{aa'}{c}\sin\gamma=0.$$

In these equations $\varphi'-\varphi=\gamma$, and $\dfrac{d^2\varphi'}{dt^2}=\dfrac{d^2\varphi}{dt^2}$, by virtue of which eliminating $\dfrac{d^2\varphi}{dt^2}$, there results

$$T=\frac{cgmm'}{\sin\gamma}\cdot\frac{a\sin\varphi'-a'\sin\varphi}{ma^2+m'a'^2}.$$

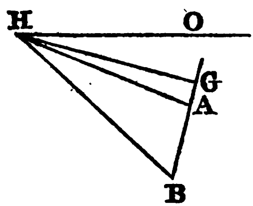
the required tension.

This expression evidently vanishes, becomes negative, and acquires maximum and minimum values corresponding to particular situations of m and m'; but such peculiarities are obvious on the slightest consideration.

The problem now resolved has long been known to mathematicians. In Simpson's Fluxions, article 182, it is formally enunciated as a lemma preparatory to finding the centre of oscillation. The result of the investigation

there given is not however intended to represent the tension of the line connecting the masses m and m', but the component of this tension in a direction perpendicular to the line a. The true value of that component is T. sin ac, or $gmm'a' \left(\frac{a \sin \phi' - a' \sin \phi}{ma^2 + m'a'^2}\right)$. Simpson assumes $g = 1$, and finds an expression equivalent to $mm'a' \left(\frac{a \sin \phi' - a' \sin \phi}{ma^2}\right)$, which is deducible from the value just found by neglecting in its denominator, the term $m'a'^2$. This term evidently arises from the inertia of the mass m'; and its omission is inconsistent with the principles of motion. The preceding expression is therefore inaccurate; and the true value above given, extended to any number of masses m', m'', m''', &c. leads us to consider the problem as inadequate to the purpose for which it was designed by its original and ingenious proposer.

SECOND SOLUTION.—*By Dr. Adrain, University of Penn. Phil.*



Let H be the point of suspension of the pendulum HA, HB, which vibrate in a vertical plane cutting the horizon in HO. On AB let fall the perpendicular HG: put HA $= a$, HB $= b$, AB $= c$, HG $= p$, angle AHO $= \alpha$, BHO $= \beta$, AHB $= \gamma$, gravity $= g$, and tension sought $=$ T, the masses at A and B being denoted by A and B. The accelerative forces with which A and B are urged in the directions of their motions downwards are $g \cos \alpha$, $g \cos \beta$, and by the resolution of forces the tension T opposes A with the accelerating force $g.\frac{\text{T}}{\text{A}}.\frac{p}{a}$ and urges B with the accelerating force $g.\frac{\text{T}}{\text{B}}.\frac{p}{b}$; and therefore the total accellerating forces acting on A and B in the directions of their motions are

$$g\left\{\cos\alpha-\frac{p\text{T}}{\text{A}a}\right\} \text{ and } g\left\{\cos\beta+\frac{p\text{T}}{\text{B}b}\right\},$$

which forces must be proportional to the distances a and b; therefore multiplying extremes and means, and dividing by g, we have

$$a\cos\beta+\frac{ap\text{T}}{\text{B}b}=b\cos\alpha-\frac{bp\text{T}}{\text{A}a}.$$

whence

$$\text{T}=\frac{\text{A}a\ .\ \text{B}b.}{\text{A}a^2+\text{B}b^2}.\frac{b\cos\alpha-a\cos\beta}{p};$$

or, which is the same,

$$\text{T}=\frac{\text{ABC}}{\text{A}a^2+\text{B}b^2}.\frac{b\cos\alpha-a\cos\beta}{\sin\gamma};$$

In the lemma given by the very ingenious English Mathematician Simpson for determining the centre of oscillations (see his Fluxions, pages 154, 155, Davis's Edition,) that author attempts the solution of the present question; but his method of investigation is erroneous. His result is that which belongs to the case in which B is infinitely greater than A, that is, when one of the masses B is in the centre of oscillation of the two bodies A and B. This appears by putting $\frac{\text{A}}{\text{B}}=0$ in the preceding value of T, by which it becomes,

$$\text{T}=\frac{\text{A}a}{bp}.(b\cos\alpha-a\cos\beta),$$

which agrees with the expression of the tension "$\text{F}\times\frac{\text{radius}}{\cos \text{C}'\text{PB}}$", as given by Simpson in the investigation of his Lemma.

To make this matter as plain as possible I shall add the following Solution:

Let Q be the centre of oscillation, SCQ$=r$, and QHO $=ce$.

The accelerating force with which Q descends is g cos ce. and therefore the accelerating force lost by A in consequence of its connection with B, is g cos Q $-$ g.

$\frac{\text{Q}}{r} \cdot \cos \text{Q}$, which acts by the lever a: but $\frac{\text{T}}{\text{A}} \cdot g$ acting by the lever, p' is the same loss of accelerative force; whence by equating these quantities, multiplying by a and p, and dividing by pg, we have

$$\text{T} = \frac{\text{A}a}{pr}(r \cos \alpha - a \cos \text{Q}).$$

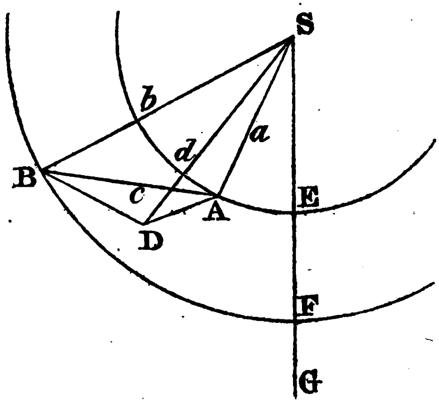
In like manner we find

$$\text{T} = \frac{\text{B}b}{pr}(-r \cos \beta + b \cos \text{Q}):$$

which values of T are equal to each other, and to the value of T in the preceding solution, as is easily shewn by substituting for cos Q and r that values in a, b, α, β.

From the first value of T, as found in this solution, the error of Simpson, both in principle and result, will be evident as well as the reason why, notwithstanding his error, he obtained the true formula for determining the centre of oscillation.

THIRD SOLUTION.—*By Professor Strong, Rutger's College, New-Brunswick, N. J.*



Let a and b denote the pendulums suspended at S in the same vertical plane, having A and B their oscillating points connected by the inflexible wire c, the pendulum rods and wire being regarded as being without mass; let A and B denote the quantities of matter in the oscillating points of a and b, and let g = 32.2 feet = the accelerating force of gravity, and SG denote its direction. Let p denote the tension of c, and put the angle ASG $= \varphi$, BSA $= c$, and dt = the constant element of the time, and suppose that the pendulums are moving towards their lowest points E, F. Then, by decomposing p and the action of gravity upon B in a direction perpendicular to SB, I find the momentum communicated to B, towards F to $=$ B$g \sin (c+\varphi)+p \sin$ SBA. Also $-\frac{d^2\varphi}{dt^2}$ denoting the increment of angular velocity of B around S, $-\frac{d^2\varphi}{dt^2}\times b$B denotes the increment of momentum received by B, which must equal the momentum communicated. Hence I derive the equation B$g \sin (c+\varphi)+p \sin$ SBA $= -b\text{B}\frac{d^2\varphi}{dt^2}$(1). In like manner, A$g \sin \varphi - p \sin$ SAB $= -a\text{A}\frac{d^2\varphi}{dt^2}$(2), p being the same here as before, but acting in a contrary direction by the law of equal action and reaction, and therefore p becomes $-p$ instead of p in the former case. Eliminating $-\frac{d^2\varphi}{dt^2}$ from (1) (2), I derive $\frac{\text{B}g\sin(c+\varphi)+p\sin \text{SBA}}{b\text{B}} = \frac{\text{A}g \sin\varphi - p \sin \text{SAB}}{a\text{A}}$, from which I find $p = \frac{\text{AB}g(b \sin \varphi - a \sin (c+\varphi))}{a\text{A} \sin \text{SBA} + b\text{B} \sin \text{SAB}}$ (3). By putting in (3) for sin SBA, sin SAB their equals $\frac{a}{c}\sin c$, $\frac{b}{c}\sin c$, it becomes $p = \frac{\text{AB}gc(b \sin \varphi - a \sin (c+\varphi))}{\sin c \times (a^2\text{A}+b^2\text{B})}$ (4).

Remark. The question here considered is in substance the same as a lemma given by T. Simpson in his Fluxions, at the 182d article. It is manifest from what has been done

above, that Simpson's result is erroneous; for he does in his investigation actually consider the weight c as infinite relatively to P, (see his figure), as may be shown from (3), given in my solution; for suppose that A is infinite, relatively to B, then (3) becomes $p = \frac{Bg(b \sin \phi - a \sin (c+\phi))}{a \sin SBA}$,

or $-p \sin SBA = Bg(\sin(c+\phi) - \frac{b}{a} \sin \phi) =$ the action of B upon A in a direction perpendicular to SB, which agrees with his result, his P corresponding to my B, and C with A, and $\sin(c+\phi) = \frac{OQ}{OP}$, $\sin \phi = \frac{\phi H}{\sigma C}$, $b =$ OP, $a =$ OC in his notation.

FOURTH SOLUTION.—*By Dr. Anderson.*

Let a and a' denote the two simple pendulums, c the inflexible wire, and x, y, x', y' the rectangular co-ordinates of the two masses m and m', originating at the point of suspension, y and y' being directed downwards. The limitations of the motion will then be expressed by the three equations

$$x^2 + y^2 - a^2 = 0,$$
$$x'^2 + y'^2 - a'^2 = 0,$$
$$(x-x')^2 + (y-y')^2 - c^2 = 0.$$

Add the variations of these equations multiplied by λ, λ', ν, to the Dynamical Equation

$$\left.\begin{array}{l} m\,d^2x\,\delta x + m\,d^2y\,\delta y - mg\delta y \\ m'd^2x'\delta x' + m'd^2y'\delta y' - m'g\delta y' \end{array}\right\} = 0,$$

equate to nought the coefficients of the variations, and we have

$$m\,d^2x + \lambda x + \nu\,(x-x') = 0,$$
$$m'd^2x' + \lambda' x' - \nu\,(x-x') = 0,$$
$$m\,d^2y + \lambda y + \nu\,(y-y') = mg,$$
$$m'd^2y' + \lambda' y' - \nu\,(y-y') = m'g;$$

where (*Lagrange, Méc. Anal.* Vol. II. p. 193) λa, $\lambda' a'$, νc represent the tensions of the lines a, a', c.

The equations of condition are satisfied by the values

$$x = a \sin (\beta + \theta),$$
$$x' = a' \sin (\beta' + \theta),$$
$$y = a \cos (\beta + \theta),$$
$$y' = a' \cos (\beta' + \theta) ;$$

where $\beta - \beta' = C$ (the angle opposite c), β or β' remaining arbitrary.

By means of these values the preceding equations furnish,

$$\frac{d^2 \theta}{dt^2} + \frac{ma \sin (\beta + \theta) + m'a' \sin (\beta' + \theta)}{ma^2 + m'a'^2} \, g = 0.$$

The numerator of the coefficient of g is equivalent to

$$\cos \theta \, (ma \sin \beta + m'a' \sin \beta'),$$
$$+ \sin \theta \, (ma \cos \beta + m'a' \cos \beta').$$

As one of the two angles β and β' is arbitrary, we may put the coefficient of $\cos \theta$ equal to nought, a simplification which, in fact, places the origin of the angle θ at the vertical of equilibrium.

The above equation then becomes

$$\frac{d^2 \theta}{dt^2} + \frac{g}{l} \, \theta = 0;$$

l being the well known expression for the length of the isochronous simple pendulum.

The value of ϖ, the required tension may be readily derived from any of the equations involving ϖ, and is equal to

$$\frac{mm'g}{ma^2 + m'a'} \cdot \frac{a' \sin (\beta + \theta) - a \sin (\beta' + \theta)}{\sin C} :$$

the angles β and β' being determined by the two equations

$$\beta - \beta' = C,$$
$$ma \sin \beta + m'a' \sin \beta' = 0.$$

From the foregoing expression it appears that the rod c comes to a given position always with the same tension, whatever be the velocity of the system as it comes to that position. This might also be inferred from the obvious circumstance that the centrifugal force does not affect the tension of the rod.

ACKNOWLEDGMENTS, &c.

The following Gentlemen favoured the Editor with Solutions to the Questions in XIX. No. 9. Vol. II. The figures to the names refer to the questions answered by each, as numbered in that article.

Dr. Bowditch, Boston ; Dr. Adrain, Phil. ; Mr. Eugenius Nulty, Phil. ; Professor Strong, New Brunswick, N J. ; Dr. Henry J. Anderson, Col. Coll., N. Y. ; Michael O'Shannessy, A. M., N. Y. ; and J. Ingersoll Bowditch ; solved all the Questions.

Messrs. James Macully, N. Y. ; Ομικρον, N. C. ; Benjamin Pierce, Junr., Cambridge University ; William Sidell, N. Y. ; and Marcus Catlin, Phil. ; solved all but 22 ; Messrs. James Docharty ; Gerardus B. Docharty, Long Island ; and Sears C. C. Walker, Phil ; solved all but 19, 22 ; Messrs John Sullivan, U. S. Navy Yard, Pensacola ; William Lenhart, York, Penn. ; S. Hammond, Patrick Lee, and John Swinburne, Brooklyn, L. I. ; solved all but the last four ; Messrs. Silas Warner, solved 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18 ; J. Thomson, 1, 2, 3, 4, 6, 8, 9, 10, 11, 12, 14, 17, 18, 20 ; John M. Wilt, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 17, 18 ; Benjamin Wiggins, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 17 ; John B. Newman, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 18 ; P. J. Rodriguez, 8, 9, 10, 11, 12, 16 ; William Vogdes, 1, 2, 3, 4, 5, 6 ; E. Lynch, 8 ; and A. H. Smith, 4.

ARTICLE XXII.

NEW QUESTIONS

To be resolved by Correspondents in No. XI.

Question I. (176.)—*By Mr. William Vogdes.*

Given $\left\{ \begin{array}{c} (x+y)^3+(x+y) = 30 \\ x-y = 1 \end{array} \right\}$ to find the values of x and y by a quadratic.

Question II. (177.)—*By Mr. John Swinburne.*

Given the sum of the squares of three numbers = 195, the sum of their cubes = 1799, and their continued product = 385 ; to find the numbers.

QUESTION III. (178.)—*By the same.*

Given $\left\{\begin{array}{l} x^8+y^6+z^4 = 146 \\ x^4y^3+x^4z^2+y^3z^2 = 89 \\ x^{12}+3x^8(y^3+z^2)+3x^4(y^3+z^2)^2 = 919 \end{array}\right\}$ to determine the values of x, y, and z.

QUESTION IV. (179.)—*By Mr. George Evans, N. Y.*

It is required to find three numbers such, that the product of the first and second, subtracted from the sum of their squares, shall be equal to 13; the product of the first and third, subtracted from the sum of their squares, shall be equal to 19; and the product of the second and third subtracted from the sum of their squares, shall be equal to 21.

QUESTION V. (180.)—*By Mr. John B. Newman.*

It is required to find how often between 1 and 300, the sum of n terms of the series 1, 2, 3, 4, 5, &c. shall be equal to that of m terms of the series 1, 3, 5, 7, 9, 11, &c.

QUESTION VI. (181.)—*By Senex, of New-York.*

The three sides of a triangular field being 3, 6, and 7 chains, now it is required to draw a right line from the greatest angle to its opposite side, so that it may divide the inscribed circle into two parts, which shall have to one another the ratio of 3 to 1; and to find the areas of the parts the triangle is divided into by the said line.

QUESTION VII. (182).—*By the same.*

On the radius of a given semicircle ABC, there is described the semicircle DEC, and from the point C there is drawn any line CEG, to cut the semicircles in E and G. Now if CEG be produced to F, so that FG may be always equal to EG; required the equation and area of the curve which is the locus of F.

QUESTION VIII. (183.)—*By Mr. Silas Warner.*

It is required to circumscribe about a given parabola, an isosceles triangle whose content shall be a minimum.

QUESTION IX. (184.)—*By Mr. P. J. Rodriguez.*

In a plane triangle ABC, given the angle A, its opposite side BC, and the rectangle of the other two sides, to find the analytical expression of the value of the angle C.

QUESTION X. (185.)—*By the same.*

To find an arc such, that its sine be half the tangent of twice that arc.

QUESTION XI. (186.)—*By Mr. Gerardus B. Docharty.*

If through a given paraboloid the diameter of the base being 8, and perpendicular altitude $10\frac{1}{2}$, a quadrilateral, three sides of which are given 3, 4, 5, moves, having its surface parallel to the base of the paraboloid and its centre of gravity coinciding with the axis. Required the solidity of the remaining part, when the area of the quadrilateral is a maximum.

QUESTION XII. (187.)—*By William Lenhart.*

It is required to find five positive numbers whose sum shall make unity; and such that, if each of the numbers be increased by unity, the respective sums shall be rational squares.

QUESTION XIII. (188.)—*By Mr. James Macully,*

It is required to inscribe in a given semispheroid the greatest possible paraboloid whose base shall be perpendicular to the conjugate axis

QUESTION XIV. (189).—*By the same.*

It is required to inscribe in a given triangle the greatest possible parabola, whose vertex shall be in one of the sides of the triangle, and its base included between the other two sides cutting them at given angles.

QUESTION XV. (190.)—*By Mr. J. Thomson, Nashville University, Tennessee.*

The integral of $\dfrac{x^2dx}{(a^2-x^2)^{\frac{1}{2}}}$ is commonly found by means of the circular *arc.* It is required to find the same by means of the circular area. Also the value of the integral, when $x = a$ and when $x = 0$.

QUESTION XVI. (191.)—*By the same.*

If the Cissoid of Diocles and the curve called the Witch, be described by the same generating circle, but having their vertices at opposite extremities of the diameter they will cut each other at a point without the circle. It is required to find the angle which these curves make with each other at this point.

QUESTION XVII. (192.)—*By Mr. William H. Sidell, N. Y.*

Let two equal parabolas, on the same side of the axis of the abscissa touch each other externally at their vertices. Now suppose one of them to roll along the arch of the other, (which remains fixed) until the terminating point of the one coincides with the same point of the other: the extremity of the leg of the moving parabola will describe a certain curve, the nature of which is required.

QUESTION XVIII. (193.)—*By Mr. E. Giddens, Fort Niagara.*

Supposing two right cones to be standing on the same

plane, being given in magnitude and position, and a given spherical ball being made to move so as to be constantly in contact with both; it is required to find the content of that part of the surface of either cone which is bounded by the line of contact.

QUESTION XIX. (194)—*By the same.*

It is required to find the sum of n terms of the series,

$$\frac{1}{\cos\phi+\cos 3\phi}+\frac{1}{\cos\phi+\cos 5\phi}+\frac{1}{\cos\phi+\cos 7\phi}+\frac{1}{\cos\phi+\cos 9\phi}+\text{, \&c.}$$

QUESTION XX. (195.)—*By* Ομικρον, *N. C.*

Given the latitude of the place, the time of observation, the bearing of a cloud, and the distance and bearing of its shadow from the station of the observer; to find its altitude.

QUESTION XXI. (196.)—*By the same.*

A cylinder being suspended by a string fixed to its side made an angle $=\theta$ with the string; but having moved the point of suspension a feet in the direction of its axis, the angle was $=\phi$ and the opposite end the highest. Required the point of suspension when the cylinder is horizontal.

QUESTION XXII. (197.)—*By M. O'Shannessy, A. M.*

A straight line and an ellipsoid of revolution being given in any manner in space, to draw a plane through this line cutting the surface so that the area of the plane surface common to both, may be given.

QUESTION XXIII. (198.)—*By the same.*

A surface of revolution of the second degree being given in magnitude and position, as well as the position of a luminous point and of the spectator's eye, to find the position of the most luminous point apparent to the eye on the given surface, and to apply the analysis in the particular case of an ellipsoid of revolution,

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QUESTION XXIV. (199.)—*By Mr. Eugenius Nulty, Phil.*

Determine the motion of a ball rolling down the surface of a hemisphere, the base of which slides by virtue of the force of the body on a perfectly smooth horizontal plane.

ERRATA AND OMMISSIONS.

For the three equations, lines 11, 13, and 16, page 57, *read*

$$\delta F' = \varrho' m \,.\, \varrho\varphi'(\cos\varphi\,\delta P - \sin\varphi\,\delta Q)\,;$$
$$\delta F' = \varrho' m \,.\, \varrho\varphi'(\cos\psi\,\delta L + \sin\varphi\psi\,\delta M)\,;$$
$$\delta F' = \varrho' m \,.\, \varrho\varphi'\,\delta\omega\,;$$

In page 58, lines 4 and 5, *dele* the expression—"neglect $c'(a'p+b'q)$, which produces terms of the second order, and."

In page 51, line 18, *dele* x.

After line 9, page 62, in Mr. Nulty's communication on rolling motion, the following has been omitted :

The equation from which the small motions of the segment of the spheroid and sphere may be determined is

$$\left.\begin{array}{l} C'\dfrac{dr}{pt}\,\delta\phi + \left\{\dfrac{d}{at}(A'\sin\omega^2\psi' + C'r\cos\omega) - m\mu c' r\omega . \omega'\right\}\delta\psi + \\ \qquad\qquad \left(A'\dfrac{d^2\omega}{dt^2} + C'r\omega\psi' - A\omega\psi'^2 - C''\omega\right)\delta\omega = 0, \end{array}\right\}$$

in which $A' = A + mc'^2$, $C' = C + m\mu c'$, and $C'' = mgc'$.

If the variations be considered as independent we shall have on neglecting quantities of the third order,

$\dfrac{dr}{dt}$ or $\dfrac{d(\phi' + \psi')}{dt} = 0\,;\ 2A'\psi' - C'r = 0\,;\ A'\dfrac{d^2\omega}{dt^2} + C'r\omega\psi' - A'\omega\psi'^2 - C''\omega = 0.$

The first and second of these equations give $r = h$

$\psi = \frac{Ch}{2A'} \cdot t$; $\phi = \left(\frac{2A' - C}{2A'}\right) ht$, and the third, on putting $k^2 = \frac{C^2r^2 - 4A'C''}{4A'^2}$ becomes $\frac{d^2\omega}{dt^2} + k^2\omega = 0$, of which the integral is $\omega = k' \cos kt$. The body revolves therefore uniformly about the principal axis c; the line of nodes projected on a horizontal plane moves equably with the velocity $\frac{Ch}{2A'}$, and the body oscillates about the vertical between the limits k' and $-k'$ in the time, $\frac{\pi}{k}$.

THE

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CONTAINING

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IN THE

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WITH

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III. Solutions to the Questions in No. XI, and new Questions and Solutions for No. XII. must arrive before the 15th of September next.

IV. No. XII. will be published on the first day of November next.

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THE

MATHEMATICAL DIARY.

No. XI.

ARTICLE XXIII.

SOLUTIONS

TO THE QUESTIONS PROPOSED IN ARTICLE XXII. NO. X.

QUESTION I. (176.)—*By Mr. Wm. Vogdes.*

Given $\left\{\begin{array}{r}(x+y)^3+(x+y)=30\\ x-y=\ 1\end{array}\right\}$ to find the values of x and y.

FIRST SOLUTION—*By Mr. John O. Sullivan, Pensacola.*

Multiplying the first equation by $x+y$, we have

$$(x+y)^4+(x+y)^2=30(x+y);$$

adding $9x^2$ to each side, we have

$$(x+y)^4+10(x+y)^2=9(x+y)^2+30(x+y):$$

completing the square, and extracting the square root, we shall get $x+y=3$, but $x-y=1$;

consequently, $x=2$, and $y=1$.*

☞ It may be proper here to observe, that a rule to solve questions of a similar nature, was *first* given in page 145 of the second New-York edition of Bonnycastle's Algebra.

SECOND SOLUTION—*By Patrick Carlin, Brooklyn.*

Assume $x+y=v$; then the first equation becomes $v^3+v=30$, multiply by v, we have $v^4+v^2=30v$; add $9v^2$ to both sides, then

$$v^4+10v^2=9v^2+30v;$$

* The solution of Plus Minus, was exactly like this.

completing the square

$$v^4+10v^2+25=9v^2+30v+25;$$

extracting the square root,

$$v^2+5=3v+5;$$

$$\therefore\ v^2=3v, \text{ or } v=3;$$

but $\qquad v=x+y;\ \therefore\ x+y=3, \text{ and } x-y=1;$

adding these two equations together

$$2x=4;\ \therefore\ x+y=3, \text{ and } y=1.$$

This solution may, with equal propriety, be ascribed to Messrs. Thomas Mooney, William Purcell, John Carmody, Brooklyn, Enoch Lansing, and J. C. Jones, Buck's county, Pa.

THIRD SOLUTION—*By Mr. Francis Sherry, N. Y.*

From the first equation

$$x^3+3xy^2+3xy^2+y^3+x+y=30;$$

from the second equation cubed, and added to its cube

$$x^3-3x^2y+3xy^2-y^3+x-y=2;$$

by addition

$$2x^3+6xy^2+2x=32,$$

or

$$x^3+3xy^2+x=16;$$

$$\therefore\ y^2=\frac{16-x^3-x}{3x}.$$

By substituting this value in the second equation, we obtain

$$x-\sqrt{\left(\frac{16-x^3-x}{3x}\right)}=1,$$

which cleared gives

$$2x^3-3x^2+2x-8=0,$$

dividing by $x-2$, there results

$$2x^2+x+4=0;$$

$$\therefore\ x=\frac{-1\pm\sqrt{-31}}{4}, \text{ and } y=\frac{-5\pm\sqrt{-31}}{4}.$$

ANOTHER SOLUTION—*By the same.*

From 2d equation $x=y+1$; hence, by substitution

$$(2y+1)^3+(2y+1)=30,$$

and, by developing, &c.

$$2y^3+3y+2-7=0;$$

dividing this equation by $y-1$, we get $2y^2+5y+7=0$; whence,

$$y=-\tfrac{5}{4}\pm\tfrac{1}{4}\sqrt{-31}. \text{ and } x=-\tfrac{1}{4}\pm\sqrt{-31}.$$

But these solutions are not *bona fide* by quadratics; since, the assumed divisors are, in fact, the roots of their respective equations.

QUESTION II. (177.)—*By Mr. John Swinburne, Brooklyn.*

Given the sum of the squares of three numbers = 195, the sum of their cubes = 1799, and their continued product = 385, to find the numbers.

FIRST SOLUTION—*By Mr. George Evans, N. Y.*

Let x, y, z, represent the numbers; put s = their sum, and $p=xy+yz+xz$: $s^2-2p=x^2+y^2+z^2=195$, and $s^3-3sp+1155$ $=x^3+y^3+z^3=1799$, or $s^3-3sp=644$; by substituting in this the value of p found from the equation $s^2-2p=195$, we shall have

$$s^3-585s=-1288\ ;$$

multiplying by s and adding $529s^2$ to both sides, we have

$$s^4-56s^2=529s^2-1288s$$

completing the square, &c. we shall get

$$s^2=23s, \text{ or } s=23;\ \text{hence } p=167\ ;$$

and consequently, $x=5$, $y=11$, and $z=7$.*

SECOND SOLUTION—*By Mr. Benjn. Wiggins, Buck's Co. Pa.*

From the question and theory of equations, it is evident that the required numbers are the three roots of a cubic equation: whence, if we let a, b, and c, represent the numbers, and f, g, and h, the co-efficients of the equation; thence, $a+b+c=-f$, $a^2+b^2+c^2=f^2-2g=195$, $a^3+b^3+c^3=-f^3+3fg-3h=1799$, and $abc=-h=385$. By substituting and transposing these, f and g are found to equal 23 and 167; whence we have, at once,

$$x^3-23x^2+167x-385=0,$$

whence x is readily to be equal 5; and, therefore,

$$\frac{x^3-23x^2+167x-385}{x-5}=x^2-18x+77.$$

Now, from the theory of equations,

$$x^2-18x+77=0. \text{ or } x^2-18x=-77\ ;$$

$$\therefore\ x=9\pm\sqrt{(81-77)}=9\pm2=11 \text{ or } 7:$$

consequently, 5, 7, and 11, are the numbers sought.

THIRD SOLUTION—*By Plus Minus, New-York.*

Let the numbers sought be represented by x, y, and z: then,

$$x^2+y^2+z^2=195,\ x^3+y^3+z^3=1799, \text{ and } xyz=385.$$

Put $x+y+z=s$, and $xy+yz+xz=r$; then by problem 75, Simp. Alg. $s^2-2r=195$, the sum of the squares of the required numbers; and $s^3-3rs+1155=1799$, the sum of their cubes: from these two equations, we readily obtain, $s^3-585s=-1288$:

---

* This solution may with equal propriety be ascribed to Professor Strong.

whence $s=23$. Now, by the first and third original equations, $x^2+y^2=195-z^2$, and $xy=\frac{385}{2}$ $\therefore$ $x+y=\sqrt{195-z^2+\frac{770}{2}}$, and $x+y+z=\sqrt{195-z^2+\frac{770}{2}}+z=23$; hence, $z^3-23z^2+167z=335$: the roots of which equation are 5, 7, and 11; which, as the unknown quantities are alike concerned, are the values of x, y, and z.

QUESTION III. (178.)—*By Mr. John Swinburne.*

Given $\left\{\begin{array}{l} x^8+y^6+z^4=146 \\ x^4y^3+x^4z^2+y^3z^2=89 \\ x^{12}+3x^8(y^3+z^2)+3x^4(y^3+z^2)^2=919 \end{array}\right\}$ to determine the values of x, y, and z.

FIRST SOLUTION—*By Mr. Francis Sherry, N. Y.*

Assuming $n=x^4$, $s=y^3+z^2$, and $p=z^2y^3$; substituting we have

$$n^2+s^2-2p=146,$$
$$ns+p=89,$$
$$n^3+3n^2s+3ns^2=919;$$

from the second of these equations, $p=89-ns$; which substitute in the first equation, there results, $n^2+s^2-178+2ns=146$, that is

$$n^2+2ns+s^2=324; \ \therefore\ n+s=18.$$

Cube this last equation and subtract the third equation from it, and

$$s^3=4913; \ \therefore\ s=17.$$

Consequently, $n=1$, but $p=89-ns=72$; hence, y^3 is readily found equal 8, and $z^2=9$. The values of x, y, and z, are 1, 2, 3, as required.

The solution of Professor Strong was similar to this.

SECOND SOLUTION—*By Mr. Marcus Catlin, Mount Holly, N. J.*

Adding twice the second equation to the first, we shall have

$$x^8+y^6+z^4+2x^4y^3+2x^4z^2+2y^3z^2=\{x^4+(y^3+z^2)\}^2=324:$$

hence, $\{x^4+(y^3+z^2)\}^3=5832$;

expanding and subtracting equation third from this, and we shall have $(y^3+z^2)^3=4913$; $\therefore$ $y^3+z^2=17$ - - - - (a).

But we have seen that

$$(x^4+y^3+z^2)^2=324; \ \therefore\ x^4+y^3+z^2=18;$$

subtracting equation a from this, and we have $x^4=1$; $\therefore$ $x=1$. Substituting this value of x in the first equation, we shall have

$$y^6+z^4=145 \quad \text{- - - - - - - - -} \quad (b).$$

But from equation (a),

$$(y^3+z^2)^2=(17)^2=289\ ;\ \therefore\ y^6+2y^3z^2+z^4=289:$$

subtracting equation (b), we shall have

$$2y^3z^2=144\ ;\ \therefore\ z^2=\frac{72}{y^3}.$$

Hence, equation (b) becomes

$$y^3+\frac{72}{y^3}=17\ ;\ \therefore\ y=2,\ z=3;\ \text{and } x=1.$$

THIRD SOLUTION—*By Mr. Silas Warner, Wrightstown, Buck's County, Pa.*

To the first equation add twice the second, and extract the square root, and

$$x^4+y^3+z^2=18\ ;\ \therefore\ y^3+z^2=18-x^4\ ;$$

this substituted in the last equation gives

$$x^{12}-54x^8+972x^4=919\ ;$$

from which we perceive that $x=1$, $\therefore\ y^3+z^2=17$, which squared, the first deducted from it, leaves $2y^3z^2=144$, or $y^3z^2=72$. Now, having their sum and product, we find that $y=2$, and $z=3$.

The solutions of Messrs. Macully, Richmond, Va. Benjamin Pierce, jun. Harvard University, and John M. Wilt, Springfield, Pa. were similar to this.

QUESTION IV. (179.)—*By Mr. George Evans, New-York.*

It is required to find three numbers, such, that the product of the first and second, subtracted from the sum of their squares, shall be equal to 13; the product of the first and third, subtracted from the sum of their squares, shall be equal to 19; and the product of the second and third, subtracted from the sum of their squares, shall be equal to 21.

FIRST SOLUTION—*By Mr. James Macully.*

Let x, y, and z, represent the numbers; then, by the conditions of the question,

$$x^2+y^2-xy=13,\ x^2+z^2-xz=19,\ y^2+z^2-yz=21.$$

Assume, $y=mx$, and $z=nx$, and the equations become

$$x^2+m^2x^2-mx^2=13,\ x^2+n^2x^2-nx^2=19,\ m^2x^2+n^2x^2-mnx=21:$$

divide equation (3) by each of the others, and we get

$$\frac{m^2+n^2-mn}{1+m^2-m}=\frac{21}{13}=a,\ \frac{m^2+n^2-mn}{1+n^2-n}=\frac{21}{19}=b\ ;$$

these are reducible to the forms

$$(1-a)m^2+n^2-mn+am-a=0,\ (1-b)n^2+m^2-mn+bn-b=0.$$

These two equations are of the second degree, in each of the quantities m and n; and therefore, by the general rules of elimi-

nation, the resulting final equation is only of the fourth degree; whence m and n are easily found to be $\frac{4}{3}$ and $\frac{5}{3}$; and, therefore,

$$x=3,\ y=4,\ \text{and}\ z=5.$$

This solution may with equal propriety be ascribed to Professor Strong, Mr. Francis Sherry, and Messrs. Gerardus B. and James Docharty.

SECOND SOLUTION—*By Mr. Benjamin Pierce.*

Let the numbers be x, ax, bx; then,

$$x^2+a^2x^2-ax^2=13,\quad x^2+b^2x^2-bx^2=19,\quad a^2x^2+b^2x^2-abx^2=21;$$

$$\therefore 21(1+a^2-a)=13(a^2+b^2-ab),\ 21(1+b^2-b)=19(a^2+b^2-ab,)$$

$$\text{or } 13b^2-13ab=9a^2-21a-21 \quad - - - - - - \quad (1),$$

$$2b^2-21b+19ab=19a^2-21 \quad - - - - - - \quad (2).$$

From twice (1) subtract 13 times (2), and we have

$$273ab-273b=265a^2-42a-315;$$

whence we obtain

$$b=\frac{265a^2-42a-315}{273(a-1)}.$$

This value of b substituted in (1), gives an equation of the 4th degree, in terms of a; whence, we find

$$a=\tfrac{4}{3},\ b=\tfrac{5}{3},\ \text{and}\ x=3:$$

consequently, the numbers are 3, 4, and 5.

QUESTION V. (180.)—*By Mr. John B. Newman.*

It is required to find how often, between 1 and 300, the sum of n terms of the series 1, 2, 3, 4, 5, &c. shall be equal to that of m terms of the series 1, 3, 5, 7, 9, 11, &c.

FIRST SOLUTION—*By Messrs. Gerardus B. Docharty and James Docharty.*

The sum of n terms of the series 1, 2, 3, 4, &c. is $\frac{n^2+n}{2}$; and that of m terms of the series 1, 3, 5, 7, 9, 11, &c. is m^2; $\therefore$ we have to find how often, between 1 and 300, $\frac{n^2+n}{2}$ is a rational square.

Make its equal $\frac{2n^2+2n}{4}=\square$; hence, $2n^2+2n=\square$; let, therefore,

$$2n^2+2n=\left(\frac{r}{s}n\right)^2=\frac{r^2}{s^2}n^2:$$

consequently,

$$n=\frac{2s^2}{r^2-2s^2};$$

by taking, $s=1, 2, 7, 12$, $r=2, 3, 10, 17$, we obtain,

$$n=1,\ 8,\ 49,\ 288,\ \text{and}\ m=1,\ 6,\ 35,\ 204;$$

$\therefore$ the number of times is 4, if $m=1$, $n=1$, be considered.

Messrs. James Macully, Enoch Lansing, J. C. Jones, and Marcus Catlin, solved this question in a similar manner.

It has been observed by Mr. Francis Sherry, that this problem is resolvable into another, viz. to find all the triangular numbers which are at the same time squares. For m terms of the series 1, 3, 5, &c. is always a square, and n terms of 1, 2, 3, &c. is always a triangular number. Taking this view of the problem, the solution will be found in the 49th art. chap. 4, part 2, of Euler's Algebra.

SECOND SOLUTION—*By Professor Strong.*

The sum of n terms of the first $=\frac{n^2+n}{2}$, and that of m terms of the second $=m^2$; $\therefore \frac{n^2+n}{2}=m^2$, or $(2n+1)^2=8m^2+1$; $\therefore 8m^2+1=\square$. Put $8m^2+1=(3m-p)^2$; $\therefore m=3p+\sqrt{(8p^2+1)}$, $p=3p'+\sqrt{(8p'^2+1)}$, and $p'=3p''+\sqrt{(8p''^2+1)}$, &c. $p^{n-1}=3p^n+\sqrt{\left(8(p^n)^2+1.\right)}$. Assume $p^n=0$; then, $p^{n-1}=1$, $p^{n-2}=6$, $p^{n-3}=35$, $p^{n-4}=204$, &c. put $m=204$; then, $(2n+1)^2=8m^2+1=(577)^2$, and $n=288$. And $m=35$ gives $n=49$, $m=6$ gives $n=8$, $m=1$ gives $n=1$; $\therefore$ the number is 4.

Otherwise,—by the second vol. Euler's Algebra, Additions by Lagrange, art. 75, I put $8m^2+1=x^2$, or $x^2-8m^2=1$; the least values of m and x are 1 and 3; $\therefore$ their greatest values are

$$m=\frac{(3+\sqrt{8})^r-(3-\sqrt{8})^r}{2},\ x=\frac{(3+\sqrt{8})^r+(3-\sqrt{8})^r}{r},$$

the greatest value of r in the present question is 4; $\therefore m=204$, $x=2n+1=577$, and $n=288$ as before, and the number of answers is 4 as above, if the values $n=1$, $m=1$, are included.

QUESTION VI. (181.)—*By Senex, New-York.*

The three sides of a triangular field being 3, 6, and 7 chains, now it is required to draw a right line from the greatest angle to its opposite side, so that it may divide the inscribed circle into two parts, which shall have to one another the ratio of 3 to 1; and to find the areas of the parts the triangle is divided into by the said line.

FIRST SOLUTION—*By Mr. O. Root, Vernon, N. Y.*

As the sides of the triangle are given, the radius of its inscribed circle is easily determined $=1.1177$; the area $=3.9246$. The circle is divided into parts which are as 3 to 1; $\therefore \frac{3.9246}{4}=.98115$ $=$ the area of the least segment; the height of which, found by the rule for circular segments, is .6662: hence, $1.1177-.6662=$

.4515 = the perpendicular from the centre on the dividing line; and as the distance from the centre to the greatest angle is found equal to 1.5; we have two sides of a rightangled-triangle to find the angles; from which the angles made by the dividing line and sides of the given triangle are determined. Whence, said line can be found; when it passes next the shortest side it is 2.556 chains; the areas are then 1.955 and 6.9892: when it passes on the other side, it is 3.087; the areas then = 4.221 and 4.273 respectively.

SECOND SOLUTION—*By a Student at Rutgers College, N.B.*

Let $2\varphi^{\circ}$ be the degrees in the arc of the lesser segment, and R° the degrees in an arc = the radius. Then I have (per question) $2\varphi^{\circ} - \sin. 2\varphi^{\circ} R^{\circ} = 90^{\circ}$; $\therefore 2\varphi^{\circ} = 132^{\circ}\ 20'\ 47''$ very nearly. The radius of the inscribed circle = 1.118 chains, and the perpendicular from its centre to the division line = 0.4517 chains; these being found, every thing else sought is easily found.

QUESTION VII. (182.)—*By the same.*

On the radius of a given semicircle ABC, there is described the semicircle DEC, and from the point C there is drawn any line CEG, to cut the semicircles in E and G. Now, if CEG be produced to F, so that FG may be always equal to EG; required the equation and area of the curve which is the locus of F.

FIRST SOLUTION—*By Mr. Benjamin Wiggins.*

Produce the given radius to F′ so that AF′ = AD: then join AG and F′F. Now, because the angle AGC is in a semicircle it is a right angle, and CG : CF : : CA : CF′. Consequently, F′FC is a right angle, and the required curve a circle, whose radius is $\frac{3}{4}$ AC; from which the area is readily found.

SECOND SOLUTION—*By Mr. Benjamin Pierce, jun.*

If $z = 2a \cos. w$ is the polar equation of ABC, the origin being at C; $z = a \cos. w$ will be that of DEC, and $EG = FG = a \cos. w$: so that the locus sought is $z = 3a \cos. w$, or the circle described on 3 times the radius of ABC as diameter.

QUESTION VIII. (183.)—*By Mr. Silas Warner.*

It is required to circumscribe about a given parabola, an iscosceles triangle whose content shall be a minimum.

FIRST SOLUTION—*By Mr. Patrick Lee, New-York.*

Let $h=$ the height of the given parabola, $y=$ the ordinate at the point of contact, x its corresponding abscissa, and, therefore, $2x$ its subtangent. From the equation of the parabola, $y=p^{\frac{1}{2}}x^{\frac{1}{2}}$; then, per similar triangles,

$$2x : p^{\frac{1}{2}}x^{\frac{1}{2}} :: h+x : \frac{hp^{\frac{1}{2}}x^{\frac{1}{2}}+p^{\frac{1}{2}}x^{\frac{3}{2}}}{2x}$$

$=$ the semibase of the triangle; the area is

$$\tfrac{1}{2}p^{\frac{1}{2}}x^{\frac{3}{2}}+hp^{\frac{1}{2}}x^{\frac{1}{2}}+\tfrac{1}{2}h^2p^{\frac{1}{2}}x^{\frac{1}{2}}=min.$$

Hence, putting the fluxion $=0$, and reducing we shall have $x=\frac{1}{3}a$; therefore, the subtangent, in this case, is equal to $\frac{2}{3}$ of the height of the parabola.

SECOND SOLUTION—*By Mr. James Macully.*

By Simpson's Fluxions, Art. 429, it appears that the least triangle that can be described about a given curve concave to its axis will be when the subtangent is half the base of the triangle. Now, if we put the abscissa of the parabola$=a$, its ordinate$=b$, we find its subtangent$=\frac{2}{3}a$, and, consequently, the perpendicular of the triangle$=\frac{4}{3}a$, and its half base $=b\sqrt{\frac{1}{3}}$, or $\frac{b}{3}\sqrt{3}$; and its area $=\frac{4ab}{9}\sqrt{3}$ as required.

QUESTION IX. (184.)—*By Mr. P. J. Rodriguez.*

In a plain triangle ABC, given the angle A, its opposite side BC, and the rectangle of the other two sides, to find the analytical expression of the value of the angle C.

FIRST SOLUTION—*By Mr. Francis Sherry.*

Put $p=$the given rectangle, $x=$one side, then $\frac{p}{x}$ will represent the other side. Now, by trigonometry, cos. $A=\dfrac{x^2+\frac{p^2}{x^2}-a^2}{2p}$.

Clearing this equation, and arranging we have

$$x^4-\left(a^2+2p(\text{cos. A})\right)x^2=-p^2.$$

Put $a^2+2p(\text{cos. A})=2s$, and complete the square we obtain

$$x=\sqrt{(s\pm\sqrt{s^2-p^2})}.$$

But the cos. $C=\dfrac{a^2+x^2-\frac{p^2}{x^2}}{2ax}$; substitute in this expression the value

found for x, and we get

$$\cos. C = \frac{a^2+s\pm\sqrt{(s^2-p^2)}-\left(\frac{p^2}{s\pm\sqrt{(s^2-p^2)}}\right)}{2a\left(\sqrt{(s\pm\sqrt{(s^2-p^2)})}\right)}.$$

SECOND SOLUTION—*By Mr. John M. Wilt.*

Imagine the perpendicular x to be drawn from B at the vertex to D in the base AC, and the triangle ABD is given in species. Let AB$=nx$, AD$=mx$, BC$=a$, and the given rectangle$=r$, then

$$\sqrt{(a^2-x^2)}+mx=\frac{r}{nx};$$

from which we derive the equation,

$$x^4-\frac{a^2n^2+rnm}{n^2+n^2m^2}x^2=-\frac{r^2}{n^2+n^2m^2}$$

a quadratic, from which we obtain x, the sine of the required angle, radius$=a$.

THIRD SOLUTION—*By Mr. Patrick Lee, New-York.*

Let BC$=a$, AB$=c$, and AC$=b$; then, per Trignometry,

$$\cos. A=\frac{b^2+c^2-a^2}{2bc};$$

freeing from fractions, we get

$$\cos. A\times 2bc=b^2+c^2-a^2, \text{ and } \cos. A\times 2bc+a^2=b^2+c^2,$$

add $2bc$ to both, and we get

$$\cos. A\times 2bc+a^2+2bc=b^2+2bc+c^2; \text{ therefore}$$
$$b+c=\sqrt{(\cos. A\times 2bc+a^2+2bc)}:$$

and in like manner, by subtracting $2bc$, we shall get

$$b-c=\sqrt{(2\cos. A\times 2bc+a^2-2bc)};$$

consequently, b and c may be easily found. Hence $\cos. A=\frac{a^2+b^2-c^2}{2ab}$, is also given.

The solutions of Messrs. Benjamin Wiggins, John O. Sullivan, and Silas Warner, were similar to this.

QUESTION X. (185.)—*By the same.*

To find an arc, such, that its sine be half the tangent of twice that arc.

FIRST SOLUTION—*By Mr. John Swinburne, Brooklyn.*

Let φ be the arc; then, by the question,

$$\sin. \varphi=\tfrac{1}{2}\text{ tang. }2\varphi.$$

But $\sin. \varphi=\frac{\tan. \varphi}{\sqrt{1+\tan.^2\varphi}}$, and $\text{tang. }2\varphi=\frac{2\text{ tang. }\varphi}{1-\tan.^2\varphi}$;

therefore, by substitution, $\frac{\text{tang.}\,\varphi}{\sqrt{1+\text{tan.}^2\varphi}}=\frac{1-\text{tan.}^2\varphi}{\text{tang.}\,\varphi}$; this equation being reduced, gives $\text{tang.}\varphi=\sqrt{3}$; whence $\varphi=60°$ or $120°$, the arc required.

Second Solution—*By Messrs. Enoch Lansing and J. C. Jones.*

Let φ represent the arc, then $\sin.\varphi=\text{tang.}\,\varphi\,\cos.\,\varphi$; and $\text{tang.}2\varphi=\frac{2\,\text{tang.}\,\varphi}{1-\text{tang.}^2\varphi}$; per ques. $\text{tang.}\,\varphi\,\cos.\,\varphi=\frac{\text{tang.}\,\varphi}{1-\text{tang.}^2\varphi}$; this gives $\sin.\,\varphi=\pm\sqrt{\tfrac{3}{4}}$: otherwise thus, ($\varphi$ as before) we have $2\sin.\,\varphi=\text{tang.}\,2\,\varphi=\frac{2\sin.\,\varphi\cos.\,\varphi}{\cos.^2\varphi-\sin.^2\varphi}$, $2\sin.\,\varphi=\frac{2\sin.\,\varphi\cos.\,\varphi}{\cos.^2\varphi-\sin.^2\varphi}$;

divide by $2\sin.\,\varphi$; we get $1=\frac{\cos.\,\varphi}{\cos.^2\varphi-\sin.^2\varphi}$, or $\cos.^2\varphi-\sin.^2\varphi$ $=\cos.\varphi$, substitute for $\sin.^2\varphi$ its equal $1-\cos.^2\varphi$, and we get

$$2\cos.^2\varphi-\cos.\varphi=1,\text{ or }\cos.^2\varphi-\tfrac{1}{2}\cos.\varphi=\tfrac{1}{2};$$

whence, $\cos.\,\varphi=1$ or $-\frac{1}{2}$, the first value gives $\varphi=0°$ or $360°$; the second gives $\varphi=120°$ or $240°$; the former of which only will give $\sin.\,\varphi=\frac{1}{2}\,\text{tang.}2\varphi$.

Question XI. (186.)—*By Mr. Gerardus B. Docharty.*

If through a given paraboloid the diameter of the base being 8, and perpendicular altitude $10\frac{1}{2}$, a quadrilateral, three sides of which are given 3, 4, 5, moves, having its surface parallel to the base of the paraboloid, and its centre of gravity coinciding with the axis. Required, the solidity of the remaining part, when the area of the quadrilateral is a maximum.

Solution—*By Professor Strong.*

When the area is a maximum, the unknown side is the diameter of a semicircle in which the quadrilateral can be inscribed, (as is proved by the writers on the elements of Isoperimetry.) Let D denote said diameter; then D can be found by the equation $D^3-500=120$, which gives $D=8.056$ nearly. The straight lines drawn from the centre of gravity to all the angles of the quadrilateral, together with the perpendiculars from it to all the sides, can now be readily found. The paraboloid will be all cut away by the figure until the side on which the less perpendicular falls first, touches its surface, thence, we must calculate the parabolic ungula cut off by this side and the other sides which cut the paraboloid until one of the angles of the figure comes into the surface; let A denote the sum of the ungulas thus found. After the angle

comes into the surface, calculate the portions of all the ungulas cut off by all the other sides, (except those which include the angle) contained between two circles on the surface of the paraboloid, one passing through the point where the angle comes into its surface, the other through the point where the angle that next enters the surface comes into it; let B denote the sum thus found.

Also, let the sides which include the angle be supposed to be produced through the angle to cut the surface, and they will cut off portions of two ungulas between the said planes; let C denote half their sum. Also, the perpendicular from the centre of gravity to the sides which contain the angle, will contain a given angle, let a denote it. They and the portions of the sides between them and the angle, will form a quadrilateral whose area is given, let D denote it; let p denote the parameter of the generating parabola of the paraboloid, and x its absciss; then, take the integral of $dx\left(\frac{apx}{2}-\mathrm{D}\right)$ between the aforesaid circles, and let F denote it: then will $\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{F}$ be the portion of the paraboloid not cut away until the second angle enters its surface. Should the second angle which enters the surface not be adjacent to the angle that first entered it, we are to proceed as before with the two angles until a third enters it, &c.

But should the two angles be adjacent, then the perpendiculars to the extreme sides contain a given angle, and with the portions of those sides and the side which falls within the paraboloid make a pentagon, with which we are to proceed as with the quadrilateral, (C denoting half the sum of the portions of the ungulas cut off by the sides on which the perpendiculars fall,) until another angle of the figure comes into the surface, &c. whatever may be the number of adjacent angles that have entered the surface, and whatever may be the number of sides of the figure and that if any given point of its surface be always upon the axis of the paraboloid and moves as in the question: and it is evident that the same process will serve for a plane figure moving down the axis of any solid of revolution, as in the question.

Question XII. (187.)—*By William Lenhart.*

It is required to find five positive numbers whose sum shall make unity; and such that, if each of the numbers be increased by unity, the respective sums shall be rational squares.

First Solution—*By the proposer.*

Let us make the question general and find three or more positive numbers whose sum shall make unity; and such that if each

of the numbers be increased by unity the respective sums shall be rational squares. If

$$\frac{2r.a+a^2}{r^2},\ \frac{2r.b+b^2}{r^2},\ \text{and}\ \frac{2r.c+c^2}{r^2},$$

represent the three numbers required, the last conditions of the question will evidently be fulfilled, and from the first we shall have

$$\frac{2r.a+a^2}{r^2}+\frac{2r.b+b^2}{r^2}+\frac{2r.c+c^2}{r^2}=1,$$

or $2r(a+b+c)+a^2+b^2+c^2=r^2$; that is, $r^2-2r(a+b+c)=a^2+b^2+c^2$. Complete the square and $r^2-2r(a+b+c)+(a+b+c)^2=a^2+b^2+c^2+(a+b+c)^2=\square$. Let now $a=n+m$, $b=n+s$, $c=n+t$, and by substitution, &c. we shall have $a^2+b^2+c^2+(a+b+c)^2=12n^2+8n(m+s+t)+m^2+s^2+t^2+(m+s+t)^2=\square$, which is evidently the case when $4n^2=m^2+s^2+t^2$. By proceeding in a similar manner for four numbers we shall find the square complete when $5n^2=m^2+s^2+t^2+v^2$. For five numbers when $6n^2=m^2+s^2+t^2+v^2+w^2$, and hence the law of progression is manifest. From what has been done it is evident, that if 4, 5, 6, &c. times any square (n^2) be divided respectively into 2 or 3, 3 or 4, 4 or 5, &c. squares (for m may be 0), which may be done readily many ways, we shall be enabled to find a variety of numbers to answer each case of the question. It is also evident that we cannot suppose, m, s, t, v, or w, &c. a negative quantity equal to, or greater than, the root (n) of the assumed square, for if we do, it will make one of the numbers either $\frac{0}{r^2}$, or negative, which will not answer.

APPLICATION.

Case I. If we take $n=7$ and divide $4n^2=4\times49=196$ into three squares whose roots are $m=-4$, $s=-6$, and $t=12$, we shall have $a=n+m=3$, $b=n+s=1$, $c=n+t=19$, $r=53$, and consequently,

$$\frac{2r.a+a^2}{r^2}=\frac{327}{2809},\ \frac{2r.b+b^2}{r^2}=\frac{107}{2809},\ \text{and}\ \frac{2r.c+c^2}{r^2}=\frac{2375}{2809}$$

the numbers required.

Case II. Take $n=3$ and divide $5n^2=5\times9=45$ into 4 squares whose roots are $m=-1$, $s=-2$, $t=2$, and $v=6$, and we shall have $a=2$, $b=1$, $c=5$, $d=9$, $r=37$, and the numbers $\frac{75}{1369}$, $\frac{152}{1369}$, $\frac{395}{1369}$, and $\frac{747}{1369}$.

Case III. This case applies to the question as published, and by taking $n=3$ and dividing $6n^2=6\times9=54$ into five squares, whose roots are $m=-1$, $s=-2$, $t=2$, $v=3$, and $w=6$, we shall have $a=2$, $b=1$, $c=5$, $d=6$, $e=9$, $r=49$, and hence the num-

bers required, $\frac{22}{2401}$, $\frac{200}{2401}$, $\frac{515}{2401}$, $\frac{624}{2401}$, and $\frac{963}{240}$, which appear to be the least that can be found.

To conclude this curious speculation, let us find eight numbers having the properties required in the question, and in order to which suppose $n=4$, $m=0$, $s=-1$, $t=-2$, $v=-3$, $w=1$, $x=2$, $y=5$, and $z=10$, then will $a=4$, $b=3$, $c=2$, $d=1$, $e=5$, $f=6$, $g=9$, $h=14$, $r=92$, and the numbers

$$\frac{185}{8464},\ \frac{372}{8464},\ \frac{561}{8464},\ \frac{752}{8464},\ \frac{945}{8464},\ \frac{1140}{8464},\ \frac{1737}{8464},\ \text{and}\ \frac{2772}{8464}:$$

or, if we suppose $n=4$, $m=-1$, $s=1$, $t=-2$, $v=2$, $w=-3$, $x=5$, $y=6$, and $z=8$, we shall have $a=3$, $b=5$, $c=2$, $d=8$, $e=1$, $f=9$, $g=10$, $h=12$, $r=100$, and hence the extraordinary numbers,

$$\frac{201}{10000},\ \frac{404}{10000},\ \frac{609}{10000},\ \frac{1025}{16000},\ \frac{1236}{10000},\ \frac{1881}{10000},\ \frac{2100}{10000},\ \text{and}\ \frac{2544}{10000}.$$

SECOND SOLUTION—*By the same.*

The general question, when stated and reduced, resolves itself into finding three or more squares, each greater than unity, whose sum shall make 4, 5, 6, &c. respectively; which may be done as follows. Let $\frac{n+a}{n-r}$, $\frac{n+a_1}{n-r}$, and $\frac{n+a_2}{n-r}$, represent the roots of the three squares required, and we shall have

$$\left(\frac{n+a}{n-r}\right)^2+\left(\frac{n+a_1}{n-r}\right)^2+\left(\frac{n+a_2}{n-r}\right)^2=4;$$

or $3n^2+2n\left(a+a_1+a_2\right)+a^2+a_1^2+a_2^2=4n^2-8nr+4r^2$. Hence, if we assume $4r^2=a^2+a_1^2+a_2^2$, we shall find $n=8r+2\left(a+a_1+a_2\right)$. By proceeding in a similar manner for four squares and assuming $5r^2=a^2+a_1^2+a_2^2+a_3^2$, we shall find $n=10r+2\left(a+a_1+a_2+a_3\right)$: for five squares and assuming $6r^2=a^2+a_1^2+a_2^2+a_3^2+a_4^2$, we shall find $n=12r+2\left(a+a_1+a_2+a_3+a_4\right)$. Hence the law of progression is manifest, and it is evident that if we divide 4, 5, 6, &c. times any square (r^2) into 2 or 3, (for a may be 0), 3 or 4, 4 or 5, &c. squares respectively, which may be done many ways, we shall be enabled to find a variety of numbers to answer the question. It is also evident, that we cannot take a, a_1, a_2, a_3, or a_4, &c. a negative quantity equal to, or greater than, the root (r) of the assumed square, because if we do it will make one of the roots either unity or less than unity, which must not be.

APPLICATION.

If we take $r=7$ and divide $4r^2=4\times 49=196$ into 3 squares, whose roots are $a=-4$, $a_1=-6$, and $a_2=12$, we shall have $n=60$, and therefore, $\frac{n+a}{n-r}=\frac{56}{53}$, $\frac{n+a_1}{n-r}=\frac{54}{53}$, and $\frac{n+a_2}{n-r}=\frac{72}{53}$, the roots of three squares whose sum shall make four and the numbers to answer the question $\frac{107}{2809}$, $\frac{327}{2809}$, and $\frac{2375}{2809}$. Again, if we take $r=3$ and divide $5r^2=5\times 9=45$ into four squares, whose roots are $a=-1$, $a_1=-2$, $a_2=2$, and $a_3=6$, we shall have $n=40$, the roots $\frac{38}{37}$, $\frac{39}{37}$, $\frac{42}{37}$, $\frac{46}{37}$, and the numbers $\frac{75}{1369}$, $\frac{152}{1369}$, $\frac{395}{1369}$, and $\frac{747}{1369}$. If we suppose $r=3$ and divide $6r^2=6\times 9=34$ (which applies to the question as published) into 5 squares, whose roots are $a=-1$, $a_1=-2$, $a_2=2$, $a_3=3$, and $a_4=6$, we shall have $n=52$, the roots $\frac{50}{49}$, $\frac{51}{49}$, $\frac{54}{49}$, $\frac{55}{49}$, $\frac{58}{49}$, and the numbers required $\frac{99}{2401}$, $\frac{200}{2401}$, $\frac{515}{2401}$, $\frac{624}{2401}$, and $\frac{963}{2401}$; which appear to be the least that can be found. If we suppose $r=3$, $a=0$, $a_1=-2$, $a_2=3$, $a_3=4$, and $a_4=5$, we shall have $n=56$ and the roots $\frac{54}{53}$, $\frac{56}{53}$, $\frac{59}{53}$, $\frac{60}{53}$, and $\frac{61}{53}$: or, if we take $r=5$ and divide $6r^2=6\times 25=150$ into 5 squares, whose roots are $a=0$, $a_1=-2$, $a_2=-3$, $a_3=-4$, and $a_4=11$, we shall have $n=64$, and thence the roots $\frac{60}{59}$, $\frac{61}{59}$, $\frac{62}{59}$, $\frac{64}{59}$, and $\frac{75}{59}$.

I shall add, by way of amusement, the following particular solution to the question as it is published in the Diary. You will readily perceive that it is derived from the preceding general solution; and that it is extremely curious.

Third Solution—*By the same.*

This question resolves itself into finding five squares, each greater than unity, whose sum shall make six, which may be done in the following manner: take six-times any square and divide it into four or five squares as often as practicable—as for example—

$$6\times 9=45=\left\{\begin{array}{l}(-1)^2+(-2)^2+(2)^2+(3)^2+(6)^2.\\ (1)^2+(-2)^2+(2)^2+(3)^2+(6)^2.\\ (-1)^2+(1)^2+(4)^2+(6)^2.\\ (-2)^2+(3)^2+(4)^2+(5)^2.\\ (2)^2+(3)^2+(4)^2+(5)^2.\end{array}\right\}$$

If now we take the roots of the first set of squares into which the

$6\times 9=74$ is divided and suppose $x-1$, $x-2$, $x+2$, $x+3$, $x+6$, respectively divided by x — the root of the assumed square (9) to be the roots of the five required squares, we shall have

$$\left(\frac{x-1}{x-3}\right)^2+\left(\frac{x-2}{x-3}\right)^2+\left(\frac{x+2}{x-3}\right)^2+\left(\frac{x+3}{x-3}\right)^2+\left(\frac{x+6}{x-3}\right)^2=6, \text{ or}$$

$5x^2+16x+54=6x^2-36x+54$, Hence, $x=52$; and therefore,

$$\frac{x-1}{x-3}=\frac{51}{49}, \frac{x-2}{x-3}=\frac{50}{49}, \frac{x+2}{x-3}=\frac{54}{49}, \frac{x+3}{x-3}=\frac{55}{49}, \text{ and } \frac{x+6}{x-3}=\frac{58}{49},$$

are the roots of 5 squares whose sum shall make six, consequently,

$$\tfrac{99}{2401}, \tfrac{200}{2401}, \tfrac{515}{2401}, \tfrac{624}{2401}, \text{ and } \tfrac{963}{2401},$$

are the numbers required.

If we take the roots of the second set of squares into which $6\times 9=54$ is divided, and suppose the required roots to be

$$\frac{x+1}{x-3}, \frac{x-2}{x-3}, \frac{x+2}{x-3}, \frac{x+3}{x-3}, \text{ and } \frac{x+6}{x-3},$$

we shall find, by proceeding as above, $x=56$, and hence the roots

$$\tfrac{54}{53}, \tfrac{57}{53}, \tfrac{58}{53}, \tfrac{59}{53}, \text{ and } \tfrac{62}{53}.$$

The third set will furnish the roots

$$\tfrac{55}{53}, \tfrac{56}{53}, \tfrac{57}{53}, \tfrac{60}{53}, \text{ and } \tfrac{62}{53}.$$

The fourth set gives

$$\tfrac{54}{53}, \tfrac{56}{53}, \tfrac{59}{53}, \tfrac{60}{53}, \text{ and } \tfrac{61}{53};$$

and the fifth

$$\tfrac{64}{61}, \tfrac{66}{61}, \tfrac{67}{61}, \tfrac{68}{61}, \text{ and } \tfrac{69}{61}.$$

In a similar manner, if we divide $6\times 25=150$ into four or five squares as often as may be, we shall be enabled to find many numbers to answer the question. One set, viz.

$6\times 25=150=(-2)^2+(-3)^2+(-4)^2+11)^2$, will furnish the roots

$$\tfrac{60}{59}, \tfrac{61}{59}, \tfrac{62}{59}, \tfrac{64}{59}, \text{ and } \tfrac{75}{59}.$$

Question XIII. (188.)—*By Mr. James Macully.*

It is required to inscribe in a given semispheroid the greatest possible paraboloid whose base shall be perpendicular to the conjugate axis.

First Solution—*By Mr. Benjamin Wiggins.*

Put a=semitransverse, b=semiconjugate diameters of the semispheroid, and let x=semibase of the paraboloid, then by the properties of the ellipse, as

$$2a : 2b :: \sqrt{(a^2-4x^2)} : \frac{b}{a}\sqrt{(a^2-4x^2)}=$$

distance from the base of the paraboloid to transverse of semi-

spheroid; and as

$$2a:2b::\sqrt{(a^2-x^2)}:\frac{b}{a}\sqrt{(a^2-x^2)}=$$

distance from the transverse to the vertex of the paraboloid, whence

$$\frac{b}{a}(\sqrt{a^2-4x^2}+\sqrt{a^2-x^2})=$$

height of the paraboloid; consequently,

$$\frac{2x^2pb}{a}\{\sqrt{(a^2-4x^2)}+\sqrt{(a^2-x^2)}\}=\text{maximum},$$

or $\sqrt{(a^2x^2-4x^4)}+\sqrt{(a^2x^2-x^4)}=$max.; whence by taking the fluxion $\frac{1}{2}(a^2x^2-4x^4)^{-\frac{1}{2}}\times(2a^2xdx-16x^3dx)+\frac{1}{2}(a^2x^2-x^4)^{-\frac{1}{2}}\times(2a^2xdx-4x^3dx)=0$; ∴

$$\frac{a^2-8x^2}{\sqrt{(a^2-4x^2)}}+\frac{a^2-2x^2}{\sqrt{(a^2-x^2)}}=0;\ \therefore\ x^6-\frac{23}{20}a^2x^4+\frac{17}{80}a^4x^2=\frac{a^6}{40};$$

from which the value of x may be found.

Second Solution—*By Mr. Silas Warner.*

Assume $b=$transverse, $2a=$conjugate of the spheroid, and $2x$ $=$the base of the required paraboloid.

Then, as $2a:b::\sqrt{4x^2-a^2}$: the distance between the ordinate or base and centre$=\frac{b}{2a}\sqrt{4x^2-a^2}=c\sqrt{4x^2-a^2}$.

Again, as $2a:b::\sqrt{a^2-x^2}$: to the distance between the ordinate which strikes the vertex of the paraboloid and the centre $=\frac{b}{2a}\sqrt{a^2-x^2}=c\sqrt{a^2-x^2}$ ∴ $c\sqrt{4x^2-a^2}+c\sqrt{(a^2-x^2)}=$height of paraboloid; and $2px^2.\overline{c\sqrt{4x^2-a^2}+c\sqrt{a^2-x^2}}=$maximum; or $\sqrt{4x^6-a^2x^4}+\sqrt{a^2x^4-x^6}=$max.; whence, by taking the differential and reducing, we obtain

$$12x^2\sqrt{a^2-x^2}+2a^2\sqrt{4x^2-a^2}=3x^2\sqrt{4x^2-a^3}+2a^2\sqrt{a^2-x^2}\ \therefore$$

$$(12x^2-2a^2)\sqrt{a^2-x^2}=(3x^2-2a^2)\sqrt{4x^2-x^2};$$

from which we know the value of x, and then the paraboloid becomes known.

Question XIV. (189.)—*By the same.*

It is required to inscribe in a given triangle the greatest possible parabola, whose vertex shall be in one of the sides of the triangle, and its base included between the other two sides cutting them at given angles.

First Solution—*By a Student at Rutgers College N.B.*

Let y be the base of the parabola, a, b, the angles at which it cuts the sides of the triangle, A, B, the corresponding angles of the triangle, C the angle opposite to y. Now, (per question) the vertex of the parabola is to touch the side opposite to C; ∴ a straight line bisecting y at right angles to said side will be a diameter of the parabola, let this perpendicular be x and x will be bisected by the curve, ∴ $\frac{1}{2}xy=$max. or $xy=$max.; let $p=$the perpendicular from C to its opposite side; then is $x=p-my$, ($m=\frac{\text{sin. A sin. } b+\text{sin. B sin. } a}{2\text{ sin. C}}$; ∴ $py-my^2=$max. which $y=\frac{p}{2m}$.

Second Solution—*By Mr. Benjamin Pierce, Student at Harvard University.*

Call h, the perpendicular, let fall from the opposite angle on the side in which the vertex of the parabola is. Let the foot of it be the origin of the co-ordinates, the perpendicular itself and the side being the axes. Represent by b and b' the segments into which the side is divided, by v the distance of the vertex of the parabola from the origin, and by x', y', and x'', y'' the points of its intersection with the other two sides; then,

$$y'=\frac{b(h-x')}{h},\quad y''=\frac{b}{h}(h-x''),$$

$$x''=\frac{(\text{tan. } e-b)x'+(b+b')h}{\text{tan. } e+b'},\quad v=\frac{y'\sqrt{x''}-y''\sqrt{x'}}{\sqrt{x''}+\sqrt{x'}}.$$

The area of the parabola is

$$\tfrac{2}{3}(y'-v)x'+\tfrac{2}{3}(y''+v)x''+\left\{\tfrac{1}{2}(y'-v)^2-\tfrac{1}{2}(y'+v)^2\right\}\frac{x'+x''}{y'+y''};$$

or by reduction

$$\tfrac{1}{6}\cdot\frac{y'+y''}{\sqrt{x'}+\sqrt{x''}}\left\{4\left(x'^{\frac{3}{2}}+x''^{\frac{3}{2}}\right)+3(\sqrt{x'}-\sqrt{x'}).(x'+x'')\right\}=\text{max.}$$

$$\text{and } y'+y''=b+b'-\frac{bx'+bx''}{h};\text{ when } e=90^\circ,$$

this expression becomes

$$\tfrac{2}{3}\cdot\frac{b+b'}{h}(h-x')x';$$

since, then $x'=x''$: this is plain a maximum when $x=\frac{1}{2}h$.

Question XV. (190.)—*By Mr. J. Thompson, Nashville University, Tennessee.*

The integral of $\frac{x^2dx}{a^2-x^2)^{\frac{1}{2}}}$ is commonly found by means of the

circular arc. It is required to find the same by means of the circular area. Also, the value of the integral, when $x=a$ and when $x=0$.

FIRST SOLUTION—*By Mr. P. J. Rodriguez.*

The formula for the integration of binomials gives

$$\int\frac{x^2dx}{(x^2-x^2)^{\frac{1}{2}}}=-\tfrac{1}{2}x(a^2-x^2)^{\frac{1}{2}}+\tfrac{1}{2}a^2\int\frac{dx}{\sqrt{(a^2-x^2)}}.$$

But by the known expression of the area of a circular, we have

$$\tfrac{1}{2}a^2\int\frac{dx}{\sqrt{(a^2-x^2)}}=s-\tfrac{1}{2}x\sqrt{(a^2-x^2)}$$

where $2a$ is the diameter of the circle, x the abscissa having the origin at the centre, and s the segment formed by the ordinate and that part of the diameter intercepted between the ordinate and the end of the arc. Substituting the last equation in the first, it comes

$$\int\frac{x^2dx}{(a^2-x^2)^{\frac{1}{2}}}=s-x\sqrt{a^2-x^2}.$$

When $x=0$, s becomes a quadrant, and when $x=a$, then $s=0$,. The second term of the second member is zero in both cases.

SECOND SOLUTION—*By Mr. Benjamin Pierce.*

Integrating by parts, we obtain

$$\int\frac{x^2dx}{\sqrt{(a^2-x^2)}}=\int(a^2-x^2)^{\frac{1}{2}}dx-x\sqrt{(a^2-x^2)}.$$

If $x=0$, the integral $=0$, and if $x=a$, the integral $=\frac{1}{4}$ of a circle.

THIRD SOLUTION—*By a Student at Rutgers College.*

Put $\sqrt{(a^2-x^2)}=y$, then

$$\frac{x^2dx}{\sqrt{(a^2-x^2)}}=-xdy=ydx-d(xy);$$

$$\therefore\int\frac{x^2dx}{\sqrt{(a^2-x^2)}}=\int ydx-xy=\int dx\sqrt{(a^2-x^2)}-x\sqrt{(a^2-x^2)}.$$

If the integral commences when $x=0$, it $=0$ when $x=0$, and it $=$ the area of a quadrant (radius$=a$) when $x=a$.

But, if the integral commences when $x=-a$, it $=$ the area of a quadrant, (rad. a) when $x=0$, and it $=$ the area of a semicircle to the same radius when $x=a$.

FOURTH SOLUTION—*By Mr. Marcus Catlin.*

Assuming $x=a-u$, $\frac{x^2dx}{\sqrt{(x^2-x^2)}}$ becomes

$$\frac{(a^2-2au+u^2)du}{\sqrt{(2au-u^2}}=\frac{-a^2du}{\sqrt{(2au-u^2)}}+\sqrt{(2au-u^2)}du.$$

But $\frac{-a^2du}{\sqrt{(2au-u^2)}}$ is reducible to

$au\sqrt{(2au-u^2)}-2a\sqrt{(2au-u^2)}du$, (Ryan's Differential and Integral Calculus, p. 224).

Hence, the given expression becomes

$$au\sqrt{(2au-u^2)}-(2a-1)\sqrt{(2au-u^2)}du,$$

whose integral is $au\sqrt{(2au-u^2)}-(2a-1)$ cir. seg. (whose radius is a and abscissa u). Or, put x for $a-u$, it will become

$$a(a-x)\sqrt{(a^2-x^2)}-(2a-1)\text{ cir. seg.}$$

(whose radius is a and abscissa $a-x)=0$, when $x=a$; and when $x=0$ it will be $a^3-(2a-1)$ (quadrant, whose radius $=a$.)

QUESTION XVI. (191.)—*By the same.*

If the Cissoid of Diocles, and the curve called the Witch, be described by the same generating circle, but having their vertices at opposite extremities of the diameter, they will cut each other at a point without the circle. It is required to find the angle which these curves make with each other at this point.

FIRST SOLUTION—*By a Student at Rutgers College.*

The equation of the cissoid is

$$y^2=\frac{x^3}{D-x}, \text{ and that of the witch is } y^2=D^2\times\left(\frac{D-x}{x^3}\right)$$

(D=the diameter of the generating circle); the vertices of these curves are placed at opposite extremities of D, as required. Now y and x are common to the curves at their point of intersection;

$$\therefore\ x=D.\frac{\sqrt{5}-1}{2},\quad y=D.\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{1}{2}}.$$

Also, by the equation of the cissoid, I have

$$\frac{dx}{dy}=\tan.\varphi=\frac{2y(D-x)}{3x^2+y^2},$$

and by the witch

$$-\frac{dx}{dy}=\tan.\theta=\frac{2xy}{D^2+y^2}.$$

Substitute for x and y their values found above, &c. and we have

$$\varphi=18^\circ\,48'\,8'',\ \theta=30^\circ\,59'\,16'';\ \therefore\ \varphi+\theta=49^\circ\,47'\,24''.$$

NOTE. It is scarcely necessary to remark, that I have supposed the origin of x to be at the vertex of the cissoid in both curves, and that x and y increase together in the cissoid; but that y decreases as x increases in the witch.

SECOND SOLUTION—*By Mr. Benjamin Pierce.*

Since the equation of the cissoid y^2 is $(a-x)-x^3=0$, and that of the witch $y^2=a^2.\frac{a-x}{x}$; we have for their point of intersection

$$\frac{x^3}{a-x}=a^2.\frac{a-x}{x};\ \text{therefore,}$$

$$x^4=a^2(a-x)^2,\ \text{or}\ x^2=a(a-x),\ \text{and}\ \therefore\ x=\frac{\sqrt{5}-1}{2}.a.$$

The tangent of the angle, that the tangent to the cissoid, at this point, makes with the axis of x is, therefore,

$$\frac{(4-\sqrt{5}).(\sqrt{5}-2)^{\frac{1}{2}}}{(3-\sqrt{5})^{\frac{3}{2}}};$$

and that which the tangent to the witch, at the same point, makes with the same axis is

$$\frac{1}{(\sqrt{5}-1).(\sqrt{5}-2)^{\frac{1}{2}}};$$

and the angle sought $=68^\circ\,54'\,28''$.

QUESTION XVII. (192.)—*By Wm. H. Sidell, N. Y.*

Let two equal parabolas, on the same side of the axis of the abscissa touch each other externally at their vertices. Now suppose one of them to roll along the arch of the other, (which remains fixed) until the terminating point of the one coincides with the same point of the other: the extremity of the leg of the moving parabola will describe a certain curve, the nature of which is required.

SOLUTION—*By Analyticus, New-Jersey.*

Let (xy), (ba), $(x'y')$, be the co-ordinates of the terminating points of the moveable and fixed curves and of their point of contact respectively, when referred to the line of the abscissas and ordinates of the fixed curve; their origin being at its vertex. It is evident that the distances of the terminating points from the point of contact are equal, and that the tangent to the curves, at the point of contact, bisects the angle formed by these distances,

and that it bisects the distance of the terminating points at right angles. From these considerations I have

$(y-y')^2+(x-x')^2=(a-y')^2+(b-x')^2$ - - - - - (1);

$\frac{y+a-2y'}{x+b-2x'}=\frac{y'}{2x'}$ - - - - - - - - - - - - - (2);

and from the nature of the curves

$px'=y'^2$, ($p=$the parameter of either curve) - (3);

$(x'y')$ being eliminated from (1) by (2) and (3) give

$$2(b-x)\times(x^2+y^2-a^2-b^2)=p(y-a)^2,$$

for the equation of the required curve.

QUESTION XVIII. (193.)—*By Mr. E. Giddens, Fort Niagara.*

Supposing two right cones to be standing on the same plane, being given in magnitude and position, and a given spherical ball being made to move so as to be constantly in contact with both; it is required to find the content of that part of the surface of either cone which is bounded by the line of contact.

SOLUTION—*By Professor Strong.*

Let k, h', be the heights of the cones, A, A', half their vertical angles, D the distance of the centres of their bases, R the radius of the ball; r, r', the radii of the circular sections of the cones passing through the corresponding points of contact; z, z', the perpendiculars from these points to the plane of the bases; $\varphi=$ the angle made by vertical planes through r and D.

Put $h+\frac{R}{\sin. A}=p$, $h'+\frac{R}{\sin. A'}=p'$, $\frac{p-p'}{\cot. A'}=a$, $\frac{\cot. A}{\cot. A'}=n$, $n^2-1=m^2$, $m^2(a^2-D^2)=b^2$, $r+R\cos. A=q$, $r'+R\cos. A'=q'$, $na-D\cos.\varphi=y$. Now from the conditions of the question, I have

$h-r\cot. A=z$, $h'-r'\cot. A'=z'$, $z+R\sin. A=z'+R\sin. A'$;

these give $nq-a=q'$ - - - - - - - - - - - - - (1),

I also have $q^2-2Dq\cos.\varphi+D^2=q'^2$ - - - - - - - - - (2),

From (1) and (2), I have

$$q=\frac{y\pm\sqrt{(y^2-b^2)}}{m^2};\ \therefore r=\frac{y-m^2R\cos. A\pm\sqrt{(y^2-b^2)}}{m^2} \quad - \quad (3);$$

this equation shows that, in general, to the same value of y there are two values of r, one of which is had by taking the radical in plus, the other in minus.

Let r'', r''', denote these values, then I have

$$\frac{r''^2-r'''^2}{2}.d\varphi=\frac{2dy.(y-m^2R\cos. A).\sqrt{(y^2-b^2)}}{m^4\{D^2-(na-y)^2\}};$$

the integral of this is to be taken between the limits, $y=-b$, and

$y=+b$, the integral thus found when multiplied cosec. A, gives the surface sought on the cone height h and vertical angle 2A. In the same manner the surface described on the other cone is found. It is scarcely necessary to remark, that the integral of the above differential is easily had by the usual methods of approximation, and that if b^2 should be a negative quantity, the line of contact will not enclose a surface.

NOTE. The locus of the ball's centre is the line of common section of two conical surfaces, one about each of the given cones; each of these surfaces being similar and similarly situated to that of the cone which it circumscribes. The height of the surface, about the cone height h, is p, and p' is the height of the other surface.

QUESTION XIX. (194.)—*By the same.*

It is required to find the sum of n terms of the series,

$$\frac{1}{\cos.\phi+\cos.3\phi}+\frac{1}{\cos.\phi+\cos.5\phi}+\frac{1}{\cos.\phi+\cos.7\phi}+\frac{1}{\cos.\phi+\cos.9\phi}+, \&c.$$

FIRST SOLUTION—*By Mr. James Macully.*

Since $\cos.a+\cos.b=\cos.\frac{1}{2}(a+b)\times\cos.\frac{1}{2}(a-b)$, we have $\cos.\phi+\cos.3\phi=2\cos.2\phi\,.\,\cos.\phi$, $\cos.\phi+\cos.5\phi=2\cos.3\phi\,.\,\cos.2\phi$, $\cos.\phi+\cos.7\phi=2\cos.4\phi\,.\,\cos.3\phi$, &c. $\therefore$ the given series becomes

$$\frac{1}{2}\left(\frac{1}{\cos.\phi\,.\,\cos.2\phi}+\frac{1}{\cos.2\phi\,.\,\cos.3\phi}+, \&c.\right)\quad \text{But}$$

$$\frac{1}{\cos.\phi}\times\frac{1}{\cos2\phi}=\sec.\phi\,.\,\sec.2\phi, \frac{1}{\cos.2\phi}\times\frac{1}{\cos.3\phi}=\sec.2\phi\,.\,\sec.3\phi, \&c.$$

therefore, the given series becomes

$$\tfrac{1}{2}\big(\sec.\phi\times\sec.2\phi+\sec.2\phi\times\sec.3\phi\ldots.+\sec.n\phi\,.\,\sec.(n+1)\phi\big).$$

Let the sum of the series be denoted by x, by adding $\frac{1}{2}$ sec. ϕ to each side of the equation, and multiplied by sin. ϕ, then it is evident, that $\frac{1}{2}\sin.\phi\,.\,\big(1\,.\,\sec.\phi+\sec.\phi\,.\,\sec.2\phi+\sec.2\phi\,.\,\sec.3\phi\ldots..+\sec.n\phi\,.\,\sec.(n+1)\phi\big)=x+\frac{1}{2}\sec.\phi)\times\sin.\,\phi$.

Now, the first part of the equation is the area of a right-angled triangle whose base is (rad. $=1$) and hypothenuse, sec. $(n+1)\phi$, $\therefore$ putting this area $=\theta$, we have

$$\theta=(x+\tfrac{1}{2}\sec.\phi)\times\sin.\phi;\ \text{and, consequently,}$$

$$x=\frac{\theta}{\sin.\phi}-\frac{1}{2}\sec.\phi=\frac{\frac{1}{2}(\sec.(n+1)\phi\times\sin.(n+1)\phi)}{\sin.\phi}-\frac{1}{2}\sec\phi=$$

$\frac{1}{2}\big(\sec.(n+1)\phi\,.\,\sin.(n+1)\phi-\sin.\phi\,.\,\sec.\phi\big)\div\sin.\phi=$ sum required; or thus, for θ substitute its equal

$$\tfrac{1}{2}\text{rad.}\times\tan.(n+1)\phi=\tfrac{1}{2}\left(\tan.(n+1)\phi\right),$$

and there results the equation

$$x=\frac{\tfrac{1}{2}\left(\tan.(n+1)\phi\right)}{\sin.\phi}-\tfrac{1}{2}\sec.\phi=\tfrac{1}{2}\left(\tan.(n+1)\phi-\sec.\phi\,.\,\sin.\phi\right)\div\sin.\phi$$

$=$ sum required.

SECOND SOLUTION—*By Mr. Benjamin Pierce.*

The nth term $=\dfrac{1}{\cos.\phi+\cos.(2n+1)\phi}=\dfrac{1}{2\cos.n\phi\,.\,\cos.(n+1)\phi}=$ $\dfrac{\tan.(n+1)\phi-\tan.n\phi}{2\sin.\phi}$. The $(n-1)$th term $=\dfrac{\tan.n\phi-\tan.(n-1)\phi}{2\sin.\phi}$;

and the sum$=\dfrac{\tan.(n+1)\phi-\tan.n\phi}{2\sin.\phi}=\dfrac{\sin.(n-1)\phi}{\sin.2\phi\cos.n\phi}$; which can be calculated by logarithms. When n is odd and $\phi=m\pi$, the sum is $=\dfrac{1}{\cos.n\phi}=\sec.n\phi=\sec.mn\pi$.

QUESTION XX. (195.)—*By* Ομικρον, *N. C.*

Given the latitude of the place, the time of observation, the bearing of a cloud, and the distance and bearing of its shadow from the station of the observer, to find its altitude.

FIRST SOLUTION—*By a Student at Rutgers College.*

The sun's altitude and bearing are easily found by the data; let A = the altitude, B = the difference of bearings of the sun and cloud, B′ = do. of the cloud and its shadow, D = the shadow's given distance; then, neglecting the earth's sphericity, I have

$$\frac{D\,.\,\tan.A\,.\,\sin.B'}{\sin.B}=\text{ the altitude required.}$$

SECOND SOLUTION—*By Mr. John M. Wilt.*

Co. lat.$\pm$dec.$=$sun's meridian altitude (a); time from noon in degrees$=d$; sin.a—(versin.d . cos.L . cos. dec.)$=$sun's true altitude (A) and 90°$\pm$dec.$=$polar dirt (P); then, sec.L . sec.A .

$$\sin.\frac{\text{co.L}+\text{co.A}+\text{P}}{2}\,.\,\sin.\frac{\text{co.L}+\text{co.A}-\text{P}}{2}=\cos.^2\tfrac{1}{2}\text{ azimuth.}$$

Wherefore we have the altitude (A) and azimuth (B); then the bearing of the cloud from the observer and the distance and bearing from the observer to the shadow being given, we have three

angles and one side, and by plane Trig. we get the length of the side (e), from the shadow to the perpendicular let fall from the cloud and as rad. : e : : tan. A : perpendicular height of the cloud.

THIRD SOLUTION—*By Mr. P. J. Rodriguez.*

Let o be the station of the observer, a the shadow of the cloud, and ob its bearing; let us suppose a plane abc perpendicular to the horizontal plane abo, so that c be the cloud, and cb its perpendicular height, the sun s will be in the straight line ac extended towards s. The figure can be readily supplied by the reader.

In the triangle abo, the angle aob is the difference between the bearing of the cloud and the bearing of its shadow; the side ao is known, and also the angle bao, since ab is the bearing of the sun from the point a. With these data the side ab will be found. Then in the right-angled triangle abc the angle cab is the altitude of the sun, which may be ascertained with the time, the latitude and the declination. With that angle and the side ab, the altitude cb will be easily found.

FOURTH SOLUTION—*By Messrs. James & Gerardus B. Docharty.*

We have from the Nautical Almanac, or an Ephemeris, the sun's altitude and declination, its bearing is easily calculated; hence the projection of the cloud's shadow upon the horizon.

Let, therefore, Z be the place of observation, L the place of the cloud, O its projection upon the horizon. (The figure can be easily conceived by the reader.) Now, in the triangle ZOL we have all the angles and the distance ZO to find OL, by plane Trigonometry. And in the right-angled triangle OPL all the angles, and the side OL being given, we readily determine PL the altitude of the cloud.

QUESTION XXI. (196.)—*By the same.*

A cylinder being suspended by a string fixed to its side made an angle $= \theta$ with the string; but having moved the point of suspension a feet in the direction of its axis, the angle was $= \phi$ and the opposite end the highest. Required the point of suspension when the cylinder is horizontal.

FIRST SOLUTION—*By Messrs. Enoch Lansing and J. C. Jones.*

It is evident that the string by which it is suspended will always pass through the centre of gravity of the body; therefore, the two lines of suspension, if produced, will intersect in the said centre. Hence, having two angles and one side of a triangle, to find

the segments into which a will be divided by a perpendicular from its opposite angle.

Let $x=$ the segment next to θ, and $y=$ the other; then,

$$\cos.\theta:\sin.\theta::x:\frac{x\sin.\theta}{\cos.\theta}=x\tan.\theta=\text{perpendicular},$$

$$\text{and }\cos.\varphi:\sin.\varphi::y:\frac{y\sin.\varphi}{\cos.\varphi}=y\tan.\varphi=\text{ do.}$$

consequently, $x:y::\tan.\varphi:\tan.\theta$, or, as $m:n$; whence, the situation of the point of suspension, when the cylinder is horizontal, is readily determined.

SECOND SOLUTION—*By Mr. Marcus Catlin.*

According to the known principles, the intersection of the two lines of suspension (when produced) is the centre of gravity of the cylinder. Now, having the three angles and one side, we shall have

$$\sin.(180^\circ-\theta-\phi):\sin.\theta::a:\frac{a\sin.\theta}{\sin.(180^\circ-\theta-\phi)}=$$

the distance of the centre of gravity from the angular point ϕ; hence we shall have,

$$1:\cos.\phi::\frac{a.\sin.\theta}{\sin.(180^\circ-\theta-\phi)}:\frac{a\sin.\theta\cos.\phi}{\sin.(180^\circ-\theta-\phi)};$$

which gives the distance of the point of suspension, from the angular point φ, when the cylinder is horizontal.

The solution of Mr. Francis Sherry, was exactly like this, having found the distance from the angular point

$$\theta=\frac{a\sin.\varphi\cos.\theta}{\sin.(180^\circ-\theta-\varphi)}.$$

THIRD SOLUTION—*By Mr. George Evans, N. Y.*

It is evident that the line of suspension, when produced, passes through the centre of the cylinder; $\therefore$ R cot. $\theta=$ the distrance sought, and $R(\cot.\theta+\cot.\varphi)=a$;

$$\therefore R\cot.\theta=\frac{a\cot.\theta}{\cot.\theta+\cot.\varphi}.$$

NOTE. If φ had been greater than θ, but the opposite end not the highest, I should have had.

$$R\cot.\theta=\frac{a\cot.\theta}{\cot.\theta-\cot.\varphi}.(R=\text{the radius of the cylinder}).$$

QUESTION XXII. (197.)—*By M. O'Shannessy, A. M.*

A straight line and an ellipsoid of revolution being given in any manner in space, to draw a plane through this line cutting the surface so that the area of the plane surface common to both, may be given.

SOLUTION—*By Professor Strong.*

I shall generalize this question by assuming the general equation of the ellipsoid.

Let $z^2+mx^2+ny^2=c^2$ - (1), denote it, and let $x=az+a'$ and $y=bz+b'$ - (2), denote the equations of the given line; assume $z+Ax+By+D=0$ - (3), for the equation of the plane, put in (3), x and y as given by (2), and there results $1+Aa+Bb=0$, $Aa'+Bb'+D=0$, (by putting the co-efficient of z in the resulting equation$=0$, since z is to be regarded as indeterminate,) these give

$$A=\frac{bD-b'}{ab'-a'b},\ B=\frac{a'-a'D}{ab'-a'b};$$

∴ all the constants in (3) are expressed in D and given quantities. I now put $A^2+m=A'$, $B^2+n=B'$, and eliminate z from (1) by (3) which gives me$A'x^2+B'y^2+2ABxy+2ADx+2BDy=C^2-D^2$,(4), which is the equation of an ellipse. Assume $x=x'+p$, $y=y'+q$, these substituted in (4) by putting the co-efficients of x' and y' each $=0$, give $A'p+ABq+AD=0$, $B'q+ABp+BD=0$; these give p and q in terms of D and given quantities.

Again, put $x'=x''\cos.\phi-y''\sin.\phi$, $y'=x''\sin.\phi+y''\cos.\phi$; then substitute these values for x' and y', in the equation previously obtained by putting the co-efficients of x' and y' each$=0$,) and put the co-efficient of the product $x''y''=0$, which gives

$$\tan.2\phi=-\frac{2AB}{B'-A'};\ \text{then put}$$

$$E=A'+B'-\sqrt{\{4A^2B^2+(B'-A')^2\}},$$
$$F=A'+B'+\sqrt{\{4A^2B^2+(B'-A')^2\}},$$
$$\frac{G^2}{2}=C'-D'-A'p^2-B'q^2-2ABpq-2ADp-2BDq;$$

and there results $Ex''^2+Fy''^2=G^2$ - - - - - - - (5), for the equation of the ellipse when referred to its axes, and $\frac{G^2}{E}$, $\frac{G^2}{F}$, are the squares of its semiaxes, and its area$=\frac{PG^2}{\sqrt{EF}}=$ (P$=$the semicircumference rad.(1)) the orthographic projection of the given area on the plane (xy): and it is evident that the expression for the area is exhibited in terms of D and known quantities. In like manner, the expressions for the projected areas on

the planes (yz) and (xz) can be found in D and given quantities. Then, by a known theorem, the sum of the squares of the projected areas $=$ the square of the given area; whence results an equation involving D and known quantities, which, solved, gives D, and thence A and B are known, and the position of the plane required to be drawn becomes known, and it passes through the given line as required.

Question XXIII. (198.)—*By the same.*

A surface of revolution of the second degree being given in magnitude and position, as well as the position of a luminous point and of the spectator's eye, to find the position of the most luminous point apparent to the eye on the given surface, and to apply the analysis in the particular case of an ellipsoid of revolution.

Solution—*By Analyticus, New-Jersey.*

If the spheroid be supposed to be perfectly smooth, so that the light comes to the eye by reflection, then it is evident that the point sought is had by describing a spheroid to touch the given surface, having the luminous point and the eye for its foci, and the point of contact will be the point required.

But if the surface be rough, so as to absorb the light and to give it out by radiation after the manner of a luminous body, then I proceed as follows, viz. I assume

$$z^2+mx^2+ny^2=k^2 \quad - - - - - - - \quad (1),$$

for the equation of the given surface, and

$(a-x^2)+(b-y)^2+(c-z)^2=r^2$, $(a'-x)^2+(b'-y)^2+(c'-z)^2=r'^2$; (abc), $a'b'c')$, being the co-ordinates of the luminous point and of the eye respectively; r, r', being the distance of those points respectively from the point of the given surface sought. Let A, A′, denote the angles which r, r', make with the tangent plane to the given surface at the point (xyz), and let the area of the given point be denoted by (1); then, I shall suppose that $\frac{\sin. A}{r^2}$ varies as the light received by the given point, and that the light received by the eye varies as $\frac{\sin. A}{r^2}\times\frac{\sin. A'}{r'^2}=\frac{\sin. A \sin. A'}{r^2r'^2}$

The equation of the tangent plane at the point (xyz) is

$$zz'+mxx'+nyy'=k^2 \quad - - - - - - \quad (2)$$

$(x'y'z')$ being the co-ordinates of any point of the tangent plane.

The perpendicular from the point (abc) is $=$

$$\frac{mxa+nyb+zc-k^2}{\sqrt{(m^2x^2+n^2y^2+z^2)}}=p, \text{ and}$$

$$\frac{mxa'+myb'+zc'-k^2}{\sqrt{(m^2x^2+n^2y^2+z^2)}}=p'=$$

to the perpendicular from $(a'b'c')$ to the tangent plane; $\therefore$ sin. $A=\frac{p}{r}$, sin. $A'=\frac{p'}{r'}$, are exhibited in terms of x, y, z, and given quantities; but r, r', are exhibited in terms of x, y, z, and given quantities; and hence the expression to be made a maximum is given in terms of x, y, z, and known quantities. The expression to be made a max. is to be subjected to (1) as an equation of condition; $\therefore$ to its differential add that of (1) multiplied by an indeterminate, then put the co-efficients of dx, dy, dz, each $=0$, and there will result three equations which will be reduced to two by eliminating the indeterminate: these two, together with (1), will serve to determine x, y, z, and the situation of the point sought becomes known.

QUESTION XXIV. (199.)—*By Mr. Eugenius Nulty, Phil.*

Determine the motion of a ball rolling down the surface of a hemisphere, the base of which slides by virtue of the force of the body on a perfectly smooth horizontal plane.

SOLUTION—*By Professor Strong.*

I suppose that the ball is a sphere, and that the densities of the ball and hemisphere are each uniform. It is hence evident that their centres are always in the same vertical plane, to which the axis of revolution of the ball is constantly at right angles. Let the line of common section of the vertical plane, with the given horizontal plane, be taken for the axis of x, and a perpendicular in the vertical plane to the axis of x be that of z; the intersection of these axes being the origin of the co-ordinates. Suppose also that m, m', are the masses of the ball and hemisphere and A the moment of inertia of the ball about its diameter, (xz) the co-ordinates of the ball's centre, $(x'o)$ those of the hemisphere's centre at any time t from the origin of the motion, and $\varphi=$the angle described by the ball about its horizontal diameter in the same time, and $dt=$the constant element of the time, and R$=$the distance between the centres of the ball and hemisphere $=$ the sum of their radii, $g=$ gravity. Then, the general formula of Dynamics becomes

$$m\left(\frac{d^2x\delta x+d^2z\delta z}{dt^2}+g\delta z\right)+\frac{m'd^2x'\delta x'}{dt^2}+\frac{\mathrm{A}d^2\varphi\delta\varphi}{dt^2}=0 \quad - \quad - \quad (1),$$

subjected to the equation of condition $(x'-x)^2+z^2=\mathrm{R}^2$ - (2). The equation of condition is satisfied by assuming $x'-x=\mathrm{R}\cos.\theta$, $z=\mathrm{R}\sin.\theta$; hence,

$$x'=x+\mathrm{R}\cos.\theta,\ \delta x'=\delta x-\mathrm{R}\delta\theta\sin.\theta,\ \text{and}\ \delta z=\mathrm{R}\delta\theta\cos.\theta.$$

I also suppose that $\delta\phi=n\delta\theta$ ($n=$const.) These values of $\delta x'$, δz, $\delta\phi$, substituted in (1) by putting the co-efficients δx, $\delta\theta$ each$=0$,

give $$\frac{md^2x}{dt^2}+\frac{m'd^2x'}{dt^2}=0, \quad - - - - - - - - (3);$$

$$m\text{R}\cos.\theta\left(\frac{d^2z}{dt^2}+g\right)-m'\text{R}\sin.\theta\frac{d^2x'}{dt^2}+\text{A}n^2\frac{d^2\theta}{dt^2}=0, \quad - - - - (4).$$

But $\frac{d^2z}{dt^2}=\text{R}.\frac{d^2\sin.\theta}{dt^2}$, $\frac{md^2x'-md^2x}{dt^2}=m\text{R}\frac{d^2\cos.\theta}{dt^2}$, or by equation (3), we have $(m+m')\frac{d^2x'}{dt^2}=m\text{R}\frac{d^2\cos.\theta}{dt^2}$; $\therefore \frac{d^2x'}{dt^2}=\frac{m}{m+m'}\times\text{R}\frac{d^2\cos.\theta}{dt^2}$.

These values of $\frac{d^2z}{dt^2}$, $\frac{d^2x'}{dt^2}$, substituted in (4), and the result multiplied by $d\theta$ and divided by $m\text{R}^2$, (by putting $\frac{m'}{m+m'}=p$, $\frac{\text{A}n^2}{m\text{R}^2}=\text{A}'$) then integrating and correcting on the supposition that $\frac{d\theta}{dt}=0$, $\theta=\theta'$ at the origin of the motion give

$$dt=-d\theta\times\left\{\frac{\text{A}'+\cos.^2\theta+p\sin.^2\theta}{\frac{2g}{\text{R}}(\sin.\theta'+\sin.\theta}\right\}^{\frac{1}{2}} \quad - - (5).$$

The integral of (5) can be found by the usual methods of approximation, whence θ is found in terms of t and known quantities, and thence, x, x', z, will be known in terms of t and known quantities.

Remarks. By putting $p=1$, or supposing m to be exceedingly small relatively to m', we have the case of a ball rolling down a hemisphere at rest. If $n=0$, the ball slides without rolling: and if $n=\frac{r'}{r}$, (r=radius of the ball and r' that of the hemisphere,) the ball rolls without sliding, in other cases it rolls and slides. It is evident that the ball will leave the hemisphere when $\frac{d^2z}{dt^2}+g=0$, for at this point the reaction of the hemisphere on the ball=0.

Now, $z=\text{R}\sin.\theta$; $\therefore \frac{dz}{dt}=\text{R}\cos.\theta\frac{d\theta}{dt}=\text{R}\cos.\theta\text{F}\theta$, and $\frac{d^2z}{dt^2}=$

$$\frac{\text{R}d(\cos.\theta\text{F}\theta)}{d\theta}\times\frac{d\theta}{dt}=\text{R}\frac{d(\cos.\theta\text{F}\theta)}{d\theta}.\ \text{F}\theta=\frac{\text{R}d(\cos.\theta\text{F}\theta)^2}{2d\,.\,\sin.\theta};\ \therefore$$

$$\frac{\text{R}d(\cos.\theta\text{F}\theta)^2}{2d\sin.\theta}+g=0, \quad - - - - - - - - (a),$$

the equation by which the value of θ being found, the point where the ball leaves the hemisphere becomes known, and it is to be observed that

$\text{F}\theta=\frac{d\theta}{dt}$ as given by (5). It is evident by (3) that the common centre of gravity of the ball and hemisphere is constantly in the same vertical line passing through its initial position.

ACKNOWLEDGMENTS, &c.

The following Gentlemen favoured the Editor with Solutions to Questions in Art. XXII. No. 10. Vol. II. The figures to the names refer to the questions answered by each, as numbered in that article.

Professor Strong, Rutger's College, New-Brunswick, New-Jersey, solved all the Questions.

Messrs. James Macully, Richmond, Virginia; Benjamin Pierce, Cambridge University; Analyticus, New-Jersey; Marcus Catlin, Mount Holly, Burlington Co. N. Jersey; and a Student at Rutgers Cellege, N. B., N. J.; solved all but 24; Messrs. Gerardus B. Docharty, and James Docharty, Long Island; solved all but 22, 24; Messrs. Enoch Lansing, and J. C. Jones, solved all but 11, 15, 17, 18, 19, 22, 23, 24; Francis Sherry, N. Y. all but 11, 13, 14, 15, 16, 17, 18, 22, 23, 24; George Evans, N. Y. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 14, 20, 21; Messrs Silas Warner, Wrightstown, and John M. Wilt, Springfield, Penn. solved 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 20; P. S. Rodriguez, Gosport, Virginia, 5, 6, 7, 8, 9, 10, 16, 20; Benjamin Wiggins, Bucks Co. Penn. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10; Messrs. P. Lee, N. Y. and John O. Sullivan, Pensacola, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13; Patrick Carlin, and Thomas Mooney, Brooklyn, 1, 2, 3, 4, 5, 8, 9, 10; William Purcell, Brooklyn, 1, 2, 3, 4, 5; O. Roof, Vernon, N. Y. 6, 9, 10; John Swinburn, Brooklyn, L. I. 2, 3, 7, 10; John Carmody, Brooklyn, 1, 4; Plus Minus, N. Y. 1, 2; William Lenhart, 12; and William Vogdes 1.

ARTICLE XXIV.

Mr. Editor,

I perceive by the last number of your Diary that Professor Strong, in investigating the vibrations of a rolling plate, obtains results differing from those which I gave in the Diary for February, 1828, while Dr. Adrain and Mr. Nulty are led by separate processes to conclusions, in all respects, coincident with my own. This difference, which Prof. Strong endeavours, not successfully, to account for, is owing to an error which he has committed in applying the analysis of Lagrange under circumstances in which that author warns the analyst against the use of his formula without the necessary modifications—namely, when the transformed variables are connected by incomplete differential equations. In such cases (and the present is one of them) Lagrange's formula cannot be employed, unless made complete by the addition of certain supplementary terms, which, for the present problem, will be found to be (in the notation of Prof. Strong)

$$R^2 \sin.\theta \frac{d\theta}{dt}\frac{d\psi}{dt}\delta\varphi - R^2 \sin.\theta \frac{d\theta}{dt}\frac{d\varphi}{dt}\delta\psi;$$

as Mr. Nulty has shown at page 91 of his solution.

With this correction the formulas of Prof. Strong will furnish the same results as those obtained by Dr. Adrain, Mr. Nulty, and myself.

H. J. Anderson.

ARTICLE XXV.

NEW QUESTIONS

TO BE RESOLVED BY CORRESPONDENTS IN NO. XII.

Question I. (200.)—*By Mr. Charles P. Otis, Colchester.*

How many words can be made with five letters of the alphabet in each word, there being 26 letters in all, and 6 vowels, admitting that a number of consonants alone will not make a word?

N. B. This question is taken from Hutton's Mathematics, vol. i. page 142, 4th New-York edition, by Dr. Adrain. As several disputes have taken place among skilful arithmeticians, respecting the correctness of the answer to this question in that work, it is respectfully submitted to the judgment of the correspondents to the Mathematical Diary.

Question II. (201.)—*By Arithmeticus, Montreal.*

Divide 140 into two such parts, that the greater being divided by the less and the less by the greater, and the less quotient being multiplied by 8, and the greater by $4\frac{1}{2}$, the two products shall be equal.

Question III. (202.)—*By Plus Minus, New-York.*

Given $\left\{\begin{array}{l} x^2+yz=16 \\ y^2+xz=17 \\ z^2+xy=22 \end{array}\right\}$ to determine the values of x, y, z, without substituting any other values for them.

Question IV. (203.)—*By X Y Z.*

Given the following equations,

$$\frac{\sqrt{x}+\sqrt{x-2}}{\sqrt{x}-\sqrt{x-2}}=\frac{2y^2}{x-2}, \quad \frac{y-\sqrt{y^2-x^2}}{y+\sqrt{y^2-x^2}}=y^2,$$

to find the values of x and y.

This question was originally proposed in No. II. of the Diary, by Thomas S. Brady, Esq.; but, by some inadvertency, the equations were different from those of the proposer, as has been remarked in page 54, vol. i.

Question V. (204.)—*By Mr. Patrick Lee, New-York.*

Given $\left\{\begin{array}{l} x=a-y^{\frac{1}{n}} \\ x^{m-1}y^{\frac{n-1}{n}}=p \end{array}\right\}$ to find x and y by a quadratic.

Question VI. (205.)—*By Mr. Silas Warner, Penn.*

The area of a right-angled triangle and the sides of a rectangle inscribed therein, being given, to construct the triangle geometrically.

Question VII. (206.)—*By Analyticus, N. Y.*

In the drawing of a lottery, according to the old system, there remained 6,300 tickets in the wheel, amongst which were two capital prizes, and were to be drawn on nine successive days, 700 tickets each day: what was the probability of drawing the two prizes on each of the nine days, and what on drawing them at all on the same day?

Question VIII. (207.)—*By Mr. E. Giddens, Rochester.*

Given the base and vertical angle of a plane triangle, and either the ratio of the inscribed circle, or the ratio of the two lines drawn from the extremities of the base to the centre of the inscribed circle, to construct the triangle.

Question IX. (208.)—*By Mr. John Swinburne.*

Given two integral cubes to determine two other integral cubes having the same difference.

Question X. (209.)—*By the same.*

In a given sector of a circle it is required to determine the greatest inscribed sector, having its angular point in the middle of the arc of the given sector.

Question XI. (210.)—*By the same.*

In a given right-angled triangle it is required to determine the greatest inscribed parabola, whose abscissa is parallel to the perpendicular of the triangle.

Question XII. (211.)—*By Mr. Patrick Lee, New-York.*

Given $x^{\frac{13}{3}}-x^{\frac{1}{3}}-4y^3+2yx^{\frac{7}{3}}=0$, to find the values of x and y, when x is a maximum.

Question XIII. (212.)—*By Mr. Francis Sherry, N. Y.*

A cone of uniform density, having the point of suspension in the circumference of the base, had the plane of the base inclined to the plane of the horizon making with it an angle ϕ. Required the solidity when the slant height is given.

QUESTION XIV. (213.)—*By Dophantus, Frederick, Md.*

It is required to find three numbers such that their sum shall make a square, and the sum of their squares a cube.

QUESTION XV. (214.)—*By Mr. O. Root.*

A cone, the diameter of whose base is 6 inches and height 3 feet, is to be wound by a thread in such a manner that it may go around once in each foot: required the length of the thread.

Although this is only a particular case of a problem that has been frequently discussed, still it will not be uninteresting to some of the contributors to the Diary.

QUESTION XVI. (215.)—*By Mr. George Evans, New-York.*

Through a given point to draw a straight line cutting an ellipse, given in magnitude and position, so that the intercepted part may be of a given length.

QUESTION XIII. (217)—*By Mr. James Macully, Richmond, Vir.*

Suppose a semicubical and a common parabola to have the same abscissa and semiordinate: it is required to find the difference of their areas; the length of the arch of the semicubical being given, and that of the common a maximum.

QUESTION XIX. (218.)—*By Mr. W. Nichols, Edinburgh, Scotl.*

Through a given point to draw a straight line cutting two given ellipses having the same centre, so that the distances of their point from the points of intersection shall be in arithmetical progression.

QUESTION XX. (219.)—*By the same.*

The same data as in the last problem, so that the lines may be in geometrical progression.

QUESTION XXI. (220.)—*By Calcul. New-York.*

Required the locus of the centres of all the circles that shall touch a given one and also a given ellipse.

QUESTION XXII. (221.)—*By Wm. Thompson, Saugerties,*

Two simple pendulums are placed near each other and vibrate in the same vertical plane. The vibrating points attract each other according to the law of gravitation. Required their motions, and particularly their small oscillations.

QUESTION XXIII. (222.)—*By Mr. Robert Gregory, Nantucket.*

A point is projected upwards along the interior surface of a hollow circular cylinder inclined to the horizon, by an impulse coinciding in direction very nearly with the straight line formed on the lower surface of the cylinder by a vertical plane passing through its axis. Required the motion.

THE

MATHEMATICAL DIARY:

CONTAINING

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IN THE

MATHEMATICS:

WITH

COLLECTIONS OF QUESTIONS

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BY EMINENT MATHEMATICIANS.

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OR THE

BEAUTIES OF THE HEAVENS DISPLAYED.

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II. New Questions must be accompanied with their Solutions; if not, they cannot be published in the Diary.

III. Solutions to the Questions in No. XII, and new Questions and Solutions for No. XIII. must arrive before the 15th of May next.

IV. No. XIII. will be published on the first day of July next.

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THE

MATHEMATICAL DIARY.

No. XII.

ARTICLE XXVI.

SOLUTIONS

TO THE QUESTIONS PROPOSED IN ARTICLE XXV. NO. XI.

QUESTION I. (200.)—*By Mr. Charles P. Otis, Colchester.*

How many words can be made with five letters of the alphabet in each word, there being 26 letters in all, and 6 vowels, admitting that a number of consonants alone will not make a word?

FIRST SOLUTION—*By Mr. Marcus Catlin, Elizabethtown.*

By known principles, the number of words that can be made with five letters in each word, admitting that a number of consonants can make a word, is

$$26\times25\times24\times23\times22=7893600 \quad . \quad . \quad . \quad (a).$$

Also, the number of words which contain consonants only, is

$$20\times12\times18\times17\times16=1860480 \quad . \quad . \quad . \quad (b).$$

Now, it is obvious that the difference of (a) and (b) equals the number of words containing vowels; that is,

$$7893600-1860480=6033120,$$

is the number required.

NOTE. If we suppose each word to contain six letters, we shall find the number of words to be

$$26\times25\times24\times23\times22\times21=165765600,$$

admitting that a number of consonants can make a word. And the words containing the consonants only, will be

$$20\times19\times18\times17\times16\times15=27907200.$$

Hence the number of words containing vowels, is

$$165765600-27907200=137858400,$$

which is the answer in Hutton.

It has been also observed by Professor Strong, and Dr. Adrain the editor of Hutton's Mathematics, in their solutions, that the answer in Hutton's course is for six letters, the five letters being a typographical error.

SECOND SOLUTION—*By Mr. John M. Wilt.*

Multiply the number of combinations of one, two, three, and four consonants out of 20 severally by the number of combinations of four, three, two, and one vowels out of 6; to the sum of these products, add the number of combinations of 5 out of 6 vowels, and multiply the latter sum by $1\times2\times3\times3\times4\times5$, (the number of permutations,) the product will be the number required, the same letter to occur but once in the same word.

$$\frac{20}{1}\times\frac{6.5.4.3}{1.2.3.4}=300$$

$$\frac{19.20}{1.\ 2}\times\frac{6.5.4}{1.2.3}=3800$$

$$\frac{18.19.20}{1.\ 2.\ 3}\times\frac{6.5}{1.2}=17100$$

$$\frac{17.18.19.20}{1.\ 2.\ 3.\ 4}\times\frac{6}{1}=29070$$

$$\frac{6.5.4.3.2}{1.2.3.4.5}=6$$

Sum 50276

Multiplied by 120 (1.2.3.4.5)

6033120 the number required.

QUESTION II. (201.)—*By Arithmeticus, Montreal.*

Divide 140 into two such parts, that the greater being divided by the less and the less by the greater, and the less quotient being multiplied by 8, and the greater by $4\frac{1}{2}$, the two products shall be equal.

FIRST SOLUTION.—*By Arithmeticus, N. Y.*

It appears from the question, that the two quotients are in the ratio of 8 to $4\frac{1}{2}$, or as 16 to 9. Now, let a and b be any two num-

bers whatever, the ratio of the quotients, $\frac{a}{b}$ and $\frac{b}{a}$, is as a^2 to b^2; this is evident by clearing of fractions.

Hence, if a represent the greater, aud b the less of the two numbers in the question proposed, we shall have

$$a^2 : b^2 :: 16 : 9,$$

and consequently,

$$a : b :: 4 : 3;$$

so that the question is reduced to this: *Divide* 140 *into two such parts that they shall be in the ratio of* 4 *to* 3, which can be readily solved by common arithmetic, without the aid of algebra; for it requires nothing more than the *division of a number in proportional parts.* See Ryan's Pestalozzian System of Arithmetic, p. 196.

SECOND SOLUTION.—*By Jacob S. Underhill and Townsend W. Hewlett, L. I.*

Let x=the greater, then $140-x$=the less, and by the question,

$$\frac{9x}{2(140-x)}=\frac{8(140-x)}{x};$$

by clearing of fractions, we have

$$9x^2=16(140-x)^2,$$

extracting the square root,

$$3x=4(140-x); \therefore 7x=560, \text{ and } x=80:$$

whence the numbers are 80 and 60.

THIRD SOLUTION.—*By Mr. O. Root.*

Put $s=\frac{1}{2}$ sum, $d=\frac{1}{2}$ difference, then, by the question,

$$\frac{s+d}{s-d}\cdot\frac{9}{2}=\frac{s-d}{s+d}\cdot 8; \therefore$$

$$d^2-\frac{50s}{7}d=-s^2;$$

hence, the numbers are 80 and 60.

NOTE. A question similar to this is to be found in Voster's Arithmetic, under the rule of *Double False Position;* the question alluded to is, "to divide 21 into two such parts, that the greater divided by the less and the less by the greater, and the less quotient being multiplied by 5 and the greater by 125, the two products shall be equal." This question can be solved according to the rule of double position, by supposing 14 and 7, 15 and 6.

Quere. Why cannot the above question be solved in a similar manner?

EDITOR.

QUESTION III. (202.)—*By Plus Minus, New-York.*

Given $\left\{\begin{array}{l} x^2+yz=16 \\ y^2+xz=17 \\ z^2+xy=22 \end{array}\right\}$ to determine the values of x, y, z, without substituting any other values for them.

FIRST SOLUTION.—*By Mr. Frederick Furber, Cambridge.*

This is the famous question solved by Dr. Wallis, in his " Wallis Opera Omnia," in 3 vols. folio, in the College library. The equations there solved are,

$$a^2+bc=l,\ b^2+ac=m,\ c^2+ab=n;$$

the principle consists in arranging the unknown quantities into factors in two equations, and by elimination in bringing them on opposite sides of the same equation, so that they can be freed from it by division. The resulting equation, he obtains, is,

$$8a^8-20la^6+2(9ll-mn)a^4+(5lmn-7l^3-m^3-n^3)a^2+(l^2-mn)^2=0.$$

If we multiply this by 2, and write a'^2 for $2a^2$, we obtain

$$a'^8-5la'^6+(9l^2-mn)a'^4+(5lmn-7l^3-m^3-n^3)a'^2+2(l^2-mn)^2=0$$

The data here given being substituted, we obtain

$$a'^8-80a'^6+1930a'^4-14313a'^2+27848+0;$$

$$\therefore\ a'^2=8,\ \text{and}\ a=\pm\sqrt{\tfrac{a'^2}{2}}=\pm\sqrt{4}=\pm2:$$

and b & c, found from the other formulas given by Dr. Wallis, are

$$b=3,\ \text{and}\ c=4.$$

The Editor feels himself much indebted to his old friend and acquaintance, Omicron of North Carolina, for the annexed history of this curious and important question.

This question has, at various periods, engaged the attention of some of the most expert analysts in Europe, and was proposed by Col. Silas Titus to Dr. Wallis in 1662. It seems to have originated with Dr. John Pell, who designed it as a trial of the skill of the Savilian Professor, who was one of the greatest mathematicians then living. In the present advanced state of the science of Algebra, there are few who would not be able to surmount all the difficulties they might meet with in the resolution of such questions: but that Dr. Wallis considered it a difficult question, and one worthy of his attention, is demonstrably certain from the immense labour he bestowed upon it, and from the minuteness and copiousness of his solution. It is said (for I have not seen the solution) to occupy 33 folio pages in his Algebra, and must be a most interesting curiosity, as exhibiting at that day the analytical abilities of a man, who, during the latter part of the seventeenth century, was the successful antagonist, in mathematical controversies, of M. M. Fermat, Freniele, and Pascal. The Doctor's solution has been characterised with, perhaps, greater severity than justice, as "operose and inelegant:" but it

ought to be remembered, that, at that time, no universal method of exterminating quantities from equations, except by substitution, was then known; that the method of elimination, as taught by De Beaume and Hudden, had, most probably, not reached England; that, notwithstanding the great obstacles with which he had to encounter, and which, to ordinary men, would be insuperable, the Dr. by unwearied assiduity and herculean exertion obtained a solution, which, for accuracy and completeness, is certainly unparalleled, and almost inimitable. The question is solved agreeably to the terms of its present enunciation at page 257 Leybourn's edition of 'the Ladies' Diary, whence the above account is abridged. The final equation is of the 8th degree, and "coincides with that found by Dr. Wallis." Two solutions are also given in No. 2 of the Apollonius, both of which terminate in a biquadratic. The numbers answering the required condition are,

$$x=2,\ y=3,\ \text{and } z=4.$$

SECOND SOLUTION.—*By Mr. Steele.*

Given $x^2+yz=a$, $y^2+xz=b$, $z^2+xy=c$. Let $x=ny$, and $z=my$; then, by substitution, we have

$$(n^2+m)y^2=a, \text{ or } y^2=a\div(n^2+m) \quad . \quad (1);$$
$$(mn+1)y^2=b, \text{ or } y^2=b\div(mn+1) \quad . \quad (2);$$
$$(m^2+n)y^2=c, \text{ or } y^2=c\div(m^2+n) \quad . \quad (3).$$

Equating the (1) equation with the (2), we have

$$m=(bn^2-a)\div(an-b)\text{l and } y^2=(an-b)\div(n^3-1);$$

which values of m and y^2, being substituted in the (3) equation, it becomes

$$(ca-b)n^4-(cb+a^2)n^3+4abn^2-(ca+b^2)n=a^2-cb.$$

If we take $a=16$, $b=17$, and $c=22$, we shall have $n=\frac{2}{3}$, $m=\frac{4}{3}$; whence we readily find $x=2$, $y=3$, and $z=4$.

This is one of the solutions referred to by Omicron, as given in No. 2 of the Apollonius. As several of the readers of the Mathematical Diary may not have an opportunity of seeing that valuable work, it may not be improper to give it insertion in the present publication.

EDITOR.

QUESTION IV. (203.)—*By X Y Z.*

Given the following equations,

$$\frac{\sqrt{x}+\sqrt{x-2}}{\sqrt{x}-\sqrt{x-2}}=\frac{2y^2}{x-2},\quad \frac{y-\sqrt{y^2-x^2}}{y+\sqrt{y^2-x^2}}=y^2,$$

to find the values of x and y.

FIRST SOLUTION.—*By Analyticus, New-Jersey.*

Multiplying the numerator and denominator of the first member of the first equation by $\sqrt{x}+\sqrt{(x-2)}$, and reducing, it becomes

$$x-1+\sqrt{(x^2-2x)}=\frac{2y^2}{x-2};$$

and the second may be reduced in a similar manner, to

$$y-\sqrt{(y^2-x^2)}=xy\ ;\ \therefore\ y^2=\frac{x}{2-x};$$

this, substituted in the above equation, gives

$$x-1+\sqrt{(x^2-2x)}=-\frac{2x}{(2-x)^2};$$

$\therefore$ by transposition,

$$x-1+\frac{2x}{(2-x)^2}=-\sqrt{(x^2-2x)}\ ;\ \text{and, squaring,}$$

$$x^2+1+\frac{4x^2}{(2-x)^4}-2x+\frac{4x^2}{(2-x)^2}-\frac{4x}{(2-x)^2}=x^2-2x;$$

$$\therefore\ (2-x)^4-4x(2-x)^2+4x^2=-4x^2(2-x)^2,$$

consequently, by extracting the root,

$$(2-x)^2-2x=2x(2-x)\,.\,\sqrt{-1}:$$

Hence, x and y are readily found.

Second Solution.—*By Omicron, N. C.*

From the second equation, after reduction, we obtain

$$x=\frac{2y^2}{y^2+1}\quad - \quad - \quad - \quad - \quad (\text{I}),$$

and from the second, by a similar operation,

$$x-1+(x^2-2x)^{\frac{1}{2}}=\frac{2y^2}{x-2}\quad - \quad - \quad - \quad (\text{II}).$$

Now, substitute the value of x found in (I) in (II), and we have

$$\frac{y^2-1}{y^2+1}+\frac{2y}{y^2+1}\sqrt{-1}=-y^2(y^2+1);\ \therefore$$

$$(y^3+y)^2=(1-y\sqrt{-1})^2,$$

$$\text{and } y^3+y(1+\sqrt{-1})-1=0;$$

from which y may be found, and, consequently, the value of x.

Third Solution—*By Mr. Silas Warner, Penn.*

From the second equation

$$y-\sqrt{(y^2-x^2)}=y^3+y^2\sqrt{(y^2-x^2)},\ \text{or}$$

$$y^2\sqrt{(y^2-x^2)}+\sqrt{(y^2-x^2)}=y-y^3\ ;\ \therefore\ \sqrt{(y^2-x^2)}=\frac{y-y^3}{y^2+1};$$

whence, $$y^2=\frac{x}{2-x};$$
this substituted in the first equation, we have
$$\frac{\sqrt{x}+\sqrt{(x-2)}}{\sqrt{x}-\sqrt{(x-2)}}=\frac{2x}{(2-x)(x-2)}.$$
Now, by multiplying the numerator and denominator of the first side of the equation by $\sqrt{x}-\sqrt{(x-2)}$, we obtain
$$\frac{2}{2x-2\sqrt{(x^2-2x)}-2}=\frac{2x}{(2-x)\,.\,(x-2)}$$
and this reduced, gives
$$5x^4-28x^3+60x^2-48x+16=0:$$
from which the value of x may be found, and then y is readily determined.

QUESTION V. (204.)—*By Mr. Patrick Lee, New-York.*

Given $\left\{\begin{array}{l} x=a-y^{\frac{1}{n}} \\ x^{n-1}\,y^{\frac{n-1}{n}}=p \end{array}\right\}$ to find x and y by a quadratic.

FIRST SOLUTION—*By Mr. Benjamin Wiggins.*

From the first equation, $y^{\frac{1}{n}}=a-x$, which substituted in the second, gives
$$x^{n-1}\times(a-x)^{n-1}=p\,;\ \therefore\ x(a-x)=p^{\frac{1}{n-1}},$$
and, consequently,
$$x=\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}-p^{\frac{1}{n-1}}\right)}$$
the value required.

SECOND SOLUTION—*By Tyro, Brooklyn.*

The first equation, by transposition, becomes
$$x+y^{\frac{1}{n}}=a\,;$$ the second, by evolution,
$$xy^{\frac{1}{n}}=p^{\frac{1}{n-1}}\,;$$ the former, squared, gives
$$x^2+2xy^{\frac{1}{n}}+y^{\frac{2}{n}}=a^2\,;$$
from which subtract four-times the latter, and we have
$$x^2-2xy^{\frac{1}{n}}+y^{\frac{2}{n}}=a^2-4p^{\frac{1}{n-1}}\,;$$

$$\therefore\ x - y^{\frac{1}{n}} = \pm\left(a^2 - 4p^{\frac{1}{n-1}}\right)^{\frac{1}{2}};$$

but $x + y^{\frac{1}{n}} = a$; $\therefore$ by addition and division,

$$x = \tfrac{1}{2}\left\{a \pm \left(a^2 - 4p^{\frac{1}{n-1}}\right)^{\frac{1}{2}}\right\};$$

and, consequently,

$$y = \left\{\tfrac{1}{2}a \pm \tfrac{1}{2}\left(a^2 - 4p^{\frac{1}{n-1}}\right)^{\frac{1}{2}}\right\}^n.$$

QUESTION VI. (205.)—*By Mr. Silas Warner, Penn.*

The area of a right-angled triangle and the sides of a rectangle inscribed therein, being given, to construct the triangle geometrically.

FIRST SOLUTION—*By Mr. William Vogdes, Philadelphia.*

Let $x =$ the perpendicular; put the side of the rectangle parallel to the base, equal to a, and the other side equal to b; then we shall have

$$x - b : a :: x : \frac{ax}{x-b} = \text{the base of the triangle};$$

$\therefore$ if $s =$ area, we have

$$\frac{ax}{x-b} \times \frac{x}{2} = s, \text{ and, consequently,}$$

$$x^2 - \frac{2s}{a}x = -\frac{2bs}{a};$$

from whence the remaining sides are readily found.

SECOND SOLUTION—*By Analyticus, N. J.*

Let a rectangle be constructed whose adjacent sides are equal to those of the inscribed rectangle; then the question requires a straight line to be drawn through one of the angles of the rectangle, cutting the other sides produced, so that the right-angled triangle thus formed may have a given area; which is done by the 70th problem of the Appendix to Simpson's Algebra.

It has been also observed by Messrs. N. Vernon, Omicron, and others, that this problem may be considered the same as Prob. 70. Simpson's Algebra; Prob. 42, of his Select Exercises, and Prop. 8, Book II. Leslie's Geometrical Analysis.

QUESTION VII. (206.)—*By Analyticus, N. Y.*

In the drawing of a lottery, according to the old system, there remained 6,300 tickets in the wheel, amongst which were two capital prizes, and were to be drawn on nine successive days, 700 tickets each day: what was the probability of drawing the two prizes on each of the nine days, and what on drawing them at all on the same day?

SOLUTION—*By Marcus Catlin.*

Let A and B represent the prizes; then the probability that A will be drawn on the first day, is $\frac{1}{9}$: also, the probability that B will be drawn the first day, is $\frac{1}{9}$. Therefore, the probability that both will be drawn on the same day, is $\frac{1}{81}$; and the probability that neither will be drawn on the first day, is $\frac{8}{9}\times\frac{8}{9}=\frac{64}{81}$. Hence, the probability that both will be drawn on the second day, is $\frac{1}{9}\times\frac{1}{9}\times\frac{64}{81}=\frac{1}{81}$, which also expresses the probability of the two prizes being drawn on each of the other days; therefore, the probability that both of the prizes will be drawn on the same day, is $9\times\frac{1}{81}=\frac{1}{9}$.

COR. Now, by the equation, one of three events must happen, viz. either none, one, or two prizes must be drawn; consequently, the sum of these three probabilities must equal a certainty or one. Hence, $1-\frac{64}{81}-\frac{1}{81}=\frac{16}{81}$ expresses the probability that one, and only one of the two prizes will be drawn on the first, second, &c. day.

The probability of drawing either none, or one, will be

$$\frac{64}{81}+\frac{16}{81}=\frac{80}{81}.$$

The probability of drawing either none, one, or two, will be

$$\frac{64}{81}+\frac{16}{81}+\frac{1}{81}=\frac{81}{81}=1, \text{ as it ought.}$$

The probability of drawing either one or two, will be

$$\frac{16}{81}+\frac{1}{81}=\frac{17}{81}.$$

The probability of drawing either none or two, will be

$$\frac{64}{81}+\frac{1}{81}=\frac{65}{81}.$$

It has been properly observed by Omicron, that this question appears to be only a modification of the 21st problem of De Moivre on Chances, from which it appears, that the probability of drawing one of the prizes on the first day is expressed nearly by the fraction $\frac{16}{81}$; that of drawing both on the the same day, by $\frac{1}{81}$ nearly, and, consequently, the probability of taking one or both on the first day is very nearly $\frac{17}{81}$. By similar steps we obtain the probabilities of the succeeding nine days.

QUESTION VIII. (207.)—*By Mr. E. Giddens, Rochester.*

Given the base and vertical angle of a plane triangle, and either

the radius of the inscribed-circle, or the ratio of the two lines drawn from the extremities of the base to the centre of the inscribed circle, to construct the triangle.

FIRST SOLUTION—*By Mr. Silas Warner.*

The vertical angle being given, we have the angle made by the lines drawn from the centre of the inscribed circle to the extremities of the base; for it will evidently be 180°, less half the sum of the angles at the base. Therefore, we have the base, vertical angle, and the ratio of the sides of the less triangle to construct it, which is done in Simpson's Exercises, Prob. 3. page 315. Now, doubling the angles at the base, the triangle is completed.

SECOND SOLUTION—*By Mr. Benjamin Wiggins.*

Put b=base, φ=vertical angle, and the ratio of the lines to the centre of the inscribed circle as m to a. Let x = one, then $\frac{nx}{m}$= the other of those lines; the angle which they make=180° $-\frac{180°-\varphi}{2}=\varphi'$; whence, $x^2-\frac{2nx^2}{m}\cos.\phi'+\frac{n^2x^2}{m^2}=b^2$; $\therefore$

$$x=\sqrt{\left\{\frac{b^2}{1-\frac{2n}{m}\cos.\varphi'+\frac{n^2}{m^2}}\right\}};$$

having determined x, and, consequently, $\frac{nx}{m}$, the angles formed by the base and these lines; and, therefore, the angles formed by the base and sides of the required triangle, (which angles are double of these,) whence the sides are easily determined.

QUESTION IX. (208.)—*By Mr. John Swinburne.*

Given two integral cubes to determine two other integral cubes having the same difference.

FIRST SOLUTION.—*By Analyticus, New-Jersey.*

Let a^3, b^3 be the given cubes, $a \ulcorner b$. Assume $a^3+x=(a+mv)^3$, $b^3+x=(b+nv)^3$; hence

$$x=3a^2mv+3am^2v^2+m^3v^3=3b^2nv+3bn^2v^2+n^3v^3; \text{ assume}$$

$$n=\frac{ma^2}{b^2}, \text{ then } v=-\frac{3ab^2}{m(a^3+b^3)}, \text{ and}$$

$-(a+mv)=\frac{2ab^3-a^4}{a^3+b^3}$, $-(b+nv)=\frac{2a^3b-b^4}{a^3+b^3}$; hence $\frac{2a^3b-b^4}{a^3+b^3}$, $\frac{2ab^3-a^4}{a^3+b^3}$ are the roots of two cubes whose difference $=a^3-b^3$. Note. I have not restricted the solution to integral cubes because it is evident that the answer cannot always be found in those numbers. It may be well to observe, since

$$a^3-b^3=\frac{(2a^3b-b^4)^3}{(a^3+b^3)^3}-\frac{(2ab^3-a^4)^3}{(a^3+b^3)^3},\text{ that}$$

$(a^4+ab^3)^3-(a^3b+b^4)^3=(a^3-b^3)\times(a^3+b^3)^3=(2a^3b-b^4)^3-(2ab^3-a^4)^3$; hence if the difference of two integral cubes can be changed to the form $(a^3-b^3)\times(a^3+b^3)^3$ two other integral cubes can be assigned having the same difference; or if the given cube are $(2a^3b-b^4)^3$, $(2ab^3-a^4)^3$ then two other integral cubes can be assigned having the same difference.

SECOND SOLUTION.—*By Tyro, Brooklyn.*

Let $(a+b^2)^3$ and $(a-b^2)^3$ be the given integral cubes; and $(x+1)^3$ and $(x-1)^3$ be the two required ones: then, by the question $6a^2b^2+2b^6=6x^2+2$, or $x^2=a^2b^2+\frac{1}{3}b^6-\frac{1}{3}$: it only remains to make this last expression a square, or, $9a^2b^2+3b^6-3=\square$: assume it $=(3ab+s)^2$ and the result will be $a=\frac{3b^6-s^2-3}{6bs}$: make $b=2$; then $a=\frac{189-s^2}{12}$: where any number may be substituted for s that will render $a>b^2$: assume $s=1$; then $a=\frac{47}{3}$ and $x=\frac{95}{3}$. The given cubes will be (by rejecting the common denominator) $(59)^3$ and $(35)^3$ and the required ones $(98)^3$ and $(92)^3$: assume $s=2$, and the given cubes will then be (by rejecting the common denominator) $281)^3$ and $(89)^3$, and the required ones $(410)^3$ and $(362)^3$. Assume $s=3$; the given cubes will then be $(9)^3$ and 1, and the required ones $(12)^3$ and $(10)^3$ which perhaps are the *least* numbers that will answer the conditions of the question.

Again, by assuming values for b, values of a are readily found in terms of s, where s is subject to the same restriction as above.

THIRD SOLUTION—*By Mr. Marcus Catlin.*

Let $a=$ the difference of the given cubes. Then x^3 and $(x-n)^3$ being two other integral cubes, we shall have by the question,

$3n\,x^2-3n^2\,x+n^3=a$ $\therefore$ $x=\dfrac{n\pm\sqrt{\left(\dfrac{4a-n^3}{3n}\right)}}{2}$. Now in order that x may be a whole number, the numerator $n\pm\sqrt{\left(\dfrac{4a-n^3}{3n}\right)}$ must be an even number, which will obviously be so, whatever be the value of n, provided $\sqrt{\left(\dfrac{4a-n^3}{3n}\right)}$ be a whole number. Hence the question is reduced to making $\dfrac{4a-n^3}{3n}$ or $12\,na-3n^4$, an integral square. If n' be the difference of the roots of the given cubes, n' will, by the question, be an integral value of n, which will render $12\,na-3n^4$ a square. Hence by Prob. 3. Case 4. p. 218 of Ryan's ed. of Bonnyeastle's Algebra, we may find other values of n which will render $12\,na-3n^4$, an integral square, in the particular examples in which it is possible. It is obvious, however, that for most values of a, the requisition of the question is impossible.

Note. For determining the cases in which it is possible a very convenient formula may be derived from the series 7, 19, 37, 61 92, 127, 169, &c.

QUESTION X. (209.)—*By the same.*

In a given sector of a circle it is required to determine the greatest inscribed sector, having its angular point in the middle of the arc of the given sector.

SOLUTION.—*By Professor Strong.*

Let R, r be the radii of the given and required sectors; A and φ half of their vertical angles; then $r^2\phi=$ the area sought $=$ a max. Also, $r\times$ sin. $(\phi+A)=R$ sin. A; therefore,

$$rd\phi=-2\phi dr,\ \text{sin.}\ (\phi+A)\times dr+\ \text{cos.}\ (\varphi+A)\times rd\phi=0\,;$$

hence, $\phi=\dfrac{\text{tan.}\ (\phi+A)}{2}$; whence, the sector is determined.

It has been observed by Omicron, N. C. that this is Quest. 257, vol. 2, Davis's ed. of the Gentlemen's Diary, where three solutions may be seen, the first of which gives $x=$ the distance of the vertex of the given sector from the chord of the required sector $=\dfrac{a^3}{a^2+b^2}$, a being equal to the given radius, and b equal to the semichord of the given sector.

Question XI. (210.)—*By the same.*

In a given right-angled triangle it is required to determine the greatest inscribed parabola, whose abscissa is parallel to the perpendicular of the triangle.

First Solution.—*By a Student of Rutgers College.*

I shall suppose the surface of the curve to touch the base of the triangle. Let y denote the straight line joining the points of intersection of the curve with the other sides of the triangle; A,B, the angles at the base; a, b, the corresponding angles at which y cuts the sides of the triangle; let p, x, x', x'', denote the perpendiculars from C, the point of intersection of y, a, b, to the base A B; and let x''' denote the abscissa to the double ordinate y. Put

$$\frac{\sin. A \sin. b}{\sin. C}=m', \frac{\sin. B \sin. a}{\sin. C}=m'', \frac{m'+m''}{2}=m, \quad m''-m'=n;$$

then, $x'=p-m'y$, $x''=p-m''y$,

$$x'''=\left(\frac{\sqrt{x'}+\sqrt{x''}}{2}\right)^2=\frac{p-my+\sqrt{\{(p-m'y).(p-m''y)\}}}{2}.$$

The area of the curve $=$

$$\tfrac{2}{3}x'''y\sqrt{(1-n^2)}=\text{a max. or } x'''y\sqrt{(1-n^2)}=\text{a max.}$$

$$\left(py-my^2+y\sqrt{\{(p-m'y).(p-m''y)\}}\right)\times\sqrt{(1-n^2)}=\text{a max.} \quad (1).$$

Also $\sin. a\sqrt{(1-\sin.^2 b)}+\sin. b\sqrt{(1-\sin.^2 a)}=\sin. C. \quad (2)$;

hence, by substituting in (1) for m, m', m'', n, their values, it will be expressed in terms of y, $\sin. a$, $\sin. b$, and given quantities; then to the differential of (1) add that of (2) multiplied by an indeterminate, and put the coefficients of dy, $d\sin. a$, $d\sin. b$, each $=o$, and eliminate the indeterminate and there will arise two equations, which with (2) will be sufficient to find y, $\sin. a$, $\sin. b$; whence every thing sought becomes known.

Cor. If a, b, are given, then

$$py-my^2+y\sqrt{\{(p-m'y).(p-m''y)\}}=\text{max.} \quad (3)$$

the differential of (3), relatively to y, (when put $=0$) gives the solution of question 14 of the last Diary; the solution there given by me being wrong, the third term of (3) having been accidentally omitted. I would observe, that the x, which is there said to be bisected by the curve, was intended to be the same with the x of the present question produced to meet the intersection of the tangent at the extremities of y.

It has been observed by Omicron, that this is Quest. 325, vol. 2, as in the note to the solution of the last question, where two solutions may be seen; and where a and b are taken to express the base and perpendicular of the triangle, x and y the coordinates of the curve and x is found $=\frac{9b}{16}$, and $y=\frac{3a}{8}$.

QUESTION XII. (211.)—*By Mr. Patrick Lee,* New-York.*

Given $x^{\frac{13}{3}}-x^{\frac{1}{3}}-4y^3+2yx^{\frac{7}{3}}=0$, to find the values of x and y, when x is a maximum.

SOLUTION.—By *Messrs. James and Gerardus B. Docharty.*

By differentiation we have

$$\tfrac{13}{3}x^{\frac{10}{3}}\,dx-\tfrac{1}{3}x^{-\frac{2}{3}}dx-12y^2dy+\tfrac{14}{3}yx^{\frac{4}{3}}dx+2x^{\frac{7}{3}}dy=0;$$

but when x is a maximum, $dx=0$;

$$\therefore\ 2x^{\frac{7}{3}}dy-12y^2dy=0,\quad \text{or } 2x^{\frac{7}{3}}=12y^2;$$

$$\text{hence; } y^2=\tfrac{1}{6}x^{\frac{7}{3}},\ \text{and } y=\sqrt{(\tfrac{1}{6}x^{\frac{3}{7}})}.$$

Consequently, by substitution, we have

$$x^{\frac{13}{3}}-x^{\frac{1}{3}}-\tfrac{2}{3}x^{\frac{7}{3}}\sqrt{(\tfrac{1}{6}x^{\frac{7}{3}})}+2x^{\frac{7}{3}}\sqrt{(\tfrac{1}{6}x^{\frac{3}{7}})}=0;$$

$$\text{or, } x^4-1-\tfrac{2}{3}x^2\sqrt{(\tfrac{1}{6}x^{\frac{3}{7}})}+2x^2\sqrt{(\tfrac{1}{6}x^{\frac{3}{7}})}=0,$$

$$\therefore\ x^4+\tfrac{4}{3}x^2\sqrt{(\tfrac{1}{6}x^{\frac{7}{3}})}-1=0.$$

From which the value of x may be obtained, and then the value of y can be readily determined.

QUESTION XIII. (212.)—*By Mr. Francis Sherry, N. Y.*

A cone of uniform density, having the point of suspension in the circumference of the base, had the plane of the base inclined to the plane of the horizon making with it an angle ϕ. Required the solidity when the slant height is given.

FIRST SOLUTION.—By *Mr. O. Root, Vernon.*

It is evident that the line of direction passing from the point of suspension through the centre of gravity, will be perpendicular to

* We are sorry to announce the death of this gentleman, which happened last harvest, after an illness of a few weeks.

the horizon, and make with the axis of the cone an angle $= \phi =$ the inclination of the base to the horizon. Put the slant height $= a$, the distance from the centre of gravity to the base measured on the axis $= x$; $\therefore 4x =$ the perpendicular height of the cone, and ϕ tang. $x =$ radius of the base; hence

$$x^2 \text{ tang.}^2\phi + 16\,x^2 = a^2, \therefore$$

$$x = \frac{a}{(\text{tang.}^2\phi + 16)^{\frac{1}{2}}},$$

whence, the solidity is equal to

$$\frac{a^3\pi \text{ tang.}^2\phi}{(\text{tang.}^2\phi + 16)^{\frac{3}{2}}}.$$

SECOND SOLUTION.—By *Analyticus, N. J.*

Let $\theta =$ the angle which the slant height makes with the plane of the base, then, since the vertical through the point of suspension passes through the centre of gravity, which is in the axis at one fourth the height of the cone from the base; I have $\tan.\theta = 4\cot.\phi$: Hence θ is known, and the slant height being given, every thing else is readily found.

THIRD SOLUTION.—By *a student of Columbia College, N.* Y.

Supposing the cone to be a right cone, let p be the point of suspension in the circumference of the base—the production of the string which is a normal to the horizon, will pass through its centre of gravity, (which is $\frac{1}{4}$ from the base of the line passing from the vertex to the centre of the base) making with this line an angle equal to the given angle φ; call the radius of the base x, the given slant side a, and the distance from the centre of gravity to the centre of the base e—by trigonometry, then we have

$$\sin.\ \varphi : x :: \cos.\ \varphi : e = \frac{x \cdot \cos.\ \varphi}{\sin.\ \varphi} = x \cot.\ \varphi.$$

The whole axis $= 4\,e = 4\,x \cdot \cot.\ \varphi$.

In the right triangle formed by the axis with the radius of the base and the slant side

$$a^2 = x^2 + 16\,x^2 \cdot \cot.\ \phi,$$

and

$$x = \sqrt{\frac{a^2}{1 + 16 \cot.^2\varphi}} = r, \text{ the radius of base.}$$

The solidity then $= \pi\, r^2 \times \frac{4}{3}\, x \cot.\ \varphi$.

Question XIV. (213.)—*By Diophantus, Frederick, Md.*

It is required to find three numbers such that their sum shall make a square, and the sum of their squares a cube.

Solutions.—By *Mr. William Lenhart.*

Solution I.—Let ax, bx and cx be the numbers, then, by the question, $ax+bx+cx=\square$ (which is evidently the case when $x=a+b+c$) and $a^2x^2+b^2x^2+c^2x^2=\text{cube}=(a^2+b^2+c^2)\times(a+b+c)^2$; or, dividing by $(a+b+c)^3$, $\dfrac{a^2+b^2+c^2}{a+b+c}=$ cube. Let now $b=na$, and $c=ma$, then will $\dfrac{a^2+b^2+c^2}{a+c+b}=\dfrac{a^2.(1+n^2+m^2)}{a.(1+n+m)}=r^3$.

$$\text{Hence, } a=\frac{r^3.(1+n+m)}{1+n^2+m^2}.$$

Now, if $n+m=n^2+m^2$, (which is well known to be the case when $n=\frac{2}{5}$ and $m=\frac{6}{5}$), then will $a=r^3$. Suppose $r=5$, then $a=125$, $b=50$, $c=150$, $x=a+b+c=325$, and thence the required numbers $ax=40625$, $bx=16250$ and $cx=48750$.

Solution II.—Let $n^2\times(m^2-1)$, $n^2\times2m^2$, and $n^2\times(m^2+1)$, represent the three numbers and the first condition of the question will be fulfilled, and from the second we shall have

$$n^4\times(6m^4+2)=\text{cube}=8n^3r^3 \text{ by assumption.}$$

$$\text{Hence, } \quad n=\frac{4r^3}{3m^4+1}.$$

Suppose $m=2$ and $r=7$, then will $n=28$ and consequently

$$n^2\times(m^2-1)=784\times3=2352,\ n^2\times2m^2=784\times8=6272 \text{ and}$$
$$n^2\times(m^2+1)=784\times5=3920\,;$$

three numbers which, on trial, will be found to answer.

Solution III.—When three numbers are in geometrical progression, the sum of their squares is always divisible by the sum of the number. Hence, if we assume

$\dfrac{x^2}{1+y+y^2}$, $\dfrac{yx^2}{1+y+y^2}$ and $\dfrac{y^2x^2}{1+y+y^2}$ for the numbers required, we shall have their sum $=x^2=\square$, and the sum of their squares $=\dfrac{x^4.(1+y^2+y^4)}{(1+y+y^2)^2}=\text{cube}=n^3x^3$, from which $x=\dfrac{n^3.(1+y+y^2)^2}{1+y^2+y^4}$, or, dividing above and below by $1+y+y^2$, according to the preceding remark, $x=\dfrac{n^3.(1+y+y^2)}{1-y+y^2}$. Now, if $y=2$ and $n=3$, then

will $x=63$; and consequently

$$\frac{x^2}{1+y+y^2}=567,\ \frac{yx^2}{1+y+y^2}=1134, \text{ and } \frac{y^2x^2}{1+y+y^2}=2268\,;$$

three numbers which will answer, and which, I presume, are the least that can be found.

Note. From a consideration of the above remark, we may solve the question thus: Let $\frac{x^2}{7}$, $\frac{2x^2}{7}$, and $\frac{4x^2}{7}$ denote the numbers,

$$\text{then } \frac{x^2}{7}+\frac{2x^2}{7}+\frac{4x^2}{7}=x^2=\square, \text{ and}$$

$\frac{x^4}{49}+\frac{4x^4}{49}+\frac{16x^4}{49}=\frac{21x^4}{49}=\frac{3x^4}{7}=$ cube; or, dividing by x^3, $\frac{3x}{7}=$ cube, which is evidently the case when $x=63$, therefore $\frac{x^2}{7}=567$, $\frac{2x^2}{7}=1134$, and $\frac{4x^2}{7}=2268$ the same as before.

Second Solution.—By *N. Vernon, Frederickton, Md.*

Let $r+s+t=m^2$, and $rx^2+sx^2+tx^2=$ the numbers sought, the first condition will be satisfied. To satisfy the second, let

$$r^2x^4+s^2x^4+t^2x^4=n^3x^3\,;\ \therefore x=\frac{n^3}{r^2+s^2+t^2}:$$

when m and n may be taken at pleasure.

Let $n=7$, $m=16$, $r=3$, $s=5$, $t=8$; then will the three numbers be $\frac{147}{4}$, $\frac{245}{4}$, and $\frac{392}{4}$, whose sum $=\frac{784}{4}=196=(14)^2$, and the sum of their squares $=\frac{117649}{8}=(\frac{49}{2})^3$.

If n be taken $=14$, then will the numbers be 2352, 3920, 6272, whole numbers, whose sum $=12544=(112)^2$, and the sum of their squares $=60236288=(392)^3$.

Third Solution.—By *a Student of Columbia College, N. Y.*

Let x^2, $3x^2$, $5x^2$, be the numbers;

then $x^2+3x^2+5x^2=9x^2$, a square, and

$$x^4+9x^4+25x^4=35x^4=n^3x^3, \text{ a cube};$$

$$\therefore x=\frac{n^3}{35}.$$

In which case $n=35$, or some multiple of 35.

FOURTH SOLUTION.—By *Mr. Marcus Catlin, N. Y.*

Let $2x^2+x$, $2x^2-2x$, and x be the three numbers. Then we shall have $2x^2+x+2x^2-2x+x=4x^2$, a square as required.

Again $(2x^2+x)^2+(2x^2-2x)^2+x^2=14x^2-4x^3$ which put equal to n^3x^3. Then $14=(n^3+4)x \therefore x=\dfrac{14}{n^3+4}$. If $n=2$, we shall have $x=\frac{7}{6} \therefore \frac{140}{36}$, $\frac{14}{36}$, and $\frac{42}{36}$ are the numbers required. A variety of numbers may be found answering the conditions of the question by assuming different values of n.

QUESTION XV. (214.)—By *Mr. O. Root.*

A cone, the diameter of whose base is 6 inches and height 3 feet, is to be wound by a thread in such a manner that it may go around once in each foot: required the length of the thread.

FIRST SOLUTION.—By *Mr. J. S. Van de Graaff, Lexington, Kentucky.*

Put a = the length of the side of the cone; b = the length of the circumference of the base; n = the interval between the threads measured on the side of the cone; v = the length of the thread after making any number of revolutions round the cone, or any part of a turn round it from the base; w = the corresponding height of the thread from the base, measured on the side of the cone; z = the corresponding length of the arch of the base. It is evident that dv and dw, form on the surface of the cone, two sides of a right angled triangle, of which the third side is $(dv^2-dn^2)^{\frac{1}{2}}$; and we therefore have,

$$a : a-w :: dz : (dv^2-dw^2)^{\frac{1}{2}}; \text{ that is,}$$

$$dv^2=dz^2\times\left(\frac{a-w}{a}\right)^2+dw^2.$$

The ratio of z to w is constant by the question; but when $z=b$, then $w=n$; therefore, $b : n : z :: w$, or $w=\dfrac{nz}{b}$. Now put $c=\dfrac{n}{b}$, $x=1-\dfrac{nz}{ab}$; and substitute these quantities in the above value of dv; we then obtain, $dv=-\dfrac{a}{c}\times dx(x^2+c^2)^{\frac{1}{2}}$. This expression may be found integrated in the Diary, page 166, vol. 1. The integral corrected to suit the present problem, is:

$$v=\frac{a}{2c}\times\left\{(c^2+1)^{\frac{1}{2}}-x(x^2+c^2)^{\frac{1}{2}}+c^2\times h.\,l\,\frac{1+(c^2+1)^{\frac{1}{2}}}{x+(x^2+c^2)^{\frac{1}{2}}}\right\}.$$

This is a general expression for the length of the thread after making any number of turns round the cone from the base to the vertex, and makes three revolutions; and consequently, in this case, $x=3b$, and $n=\frac{1}{3}a$. Substituting these values in the above general expression, we shall readily find 47.74935 inches, the required length of the thread.

SECOND SOLUTION.—By *Professor Strong, Rutgers College.*

The orthographic projection of the thread on the cone's base is evidently the spiral of Archimedes, the origin of the spiral being at the centre of the base; let r, $r+dr$ be two successive radii vectores of the spiral; $d\phi=$ the elementary angle which they make with each other; then $dr^2+r^2d\phi^2=$the square of the element of the spiral; let $n=$ the tangent of the angle which the side of the cone makes with the plane of its base, $x=$ the perpendicular to the base through the extremity of r, limited by the conical surface; then $dx=-ndr$, let $ds=$ the element of the thread; then

$$ds=\sqrt{r^2d\phi^2+(1+n^2)dr^2} \quad - \quad - \quad - \quad (1).$$

Put $p=3.14159$*etc.*; $a=$ the distance between the successive gyrations of the thread taken on the slant height of the cone; then by the nature of the spiral I have $d\phi=\dfrac{2p\sqrt{1+n^2}}{a}\times dr$; hence by (1)

$$ds=-\frac{2p}{a}\sqrt{1+n^2}\times\sqrt{\frac{a^2}{4p^2}+r^2}\times dr \quad - \quad (2);$$

(2) suppose s to commence at the base of the cone, since the sign $-$ is written before the right hand side of the equation, or which comes to the same thing, it supposes r to decrease when s increases.

By putting $\dfrac{a}{2p}=b$, R $=$ the radius of the base of the cone, and $r=$ the radius of a circular section of the cone at the variable extremity of s,; I have by integration and correction on the supposition that s, commences at the base of the cone

$$s=\sqrt{1+n^2}\times$$
$$\left(\frac{\mathrm{R}\times(\mathrm{R}^2+b^2)^{\frac{1}{2}}-r(r^2+b^2)^{\frac{1}{2}}}{2b}+\frac{b}{2}\text{hy. log.}\left(\frac{\mathrm{R}+(\mathrm{R}^2+b^2)^{\frac{1}{2}}}{r+(r^2+b^2)^{\frac{1}{2}}}\right)\right) \quad (3);$$

by putting in (3) $r=0$, I have

$$s'=\sqrt{1+n^2}\times\left(\frac{\mathrm{R}(\mathrm{R}^2+b^2)^{\frac{1}{2}}}{2b}+\frac{b}{2}\text{hy. log.}\left(\frac{\mathrm{R}+(\mathrm{R}^2+b^2)^{\frac{1}{2}}}{b}\right)\right) \quad (4);$$

$r = c \cdot \varphi$

$s' =$ the whole length of the thread sought. In this question by putting $a = 12$ inches, I find $s' = 47.816$ inches nearly.

THIRD SOLUTION.—By *Mr. Marcus Catlin, N. Y.*

Let ABF* be the surface of the cone spread out on a plane. from A draw in such a manner that A*d* shall vary as the angle BA*d* and that AD shall = one foot. Then will A*d*D, be equal to the thread wound once around the cone. Now if $n =$ the slant height of the cone in feet, the thread will go around n times. Therefore if BFCG$=n$ times BF and the curve AD be continued to C, the curve ADEC will equal the whole length of the thread. But since the curve ADC varies as the radius vector AD, it is the spiral of Archimedes.

Put $t =$ the variable angle BA*d*, and $s =$ the curve A*d*: then we shall have $n = at$, a being a constant known quantity, and n the radius vector. Also (Ryan's Cal. Art, 267) we shall have

$$ds = adt\sqrt{(t^2+1)}$$

$$\text{Hence } s = \frac{a}{2}\sqrt{(1+t^2)} + \frac{a}{8} \times \text{ hyp. log.} \{t + \sqrt{(1+t^2)}\} + \text{C}.$$

Taking the limits and reducing and we shall have the value of s.

Cor. If $r =$ the radius of the base, and the thread be wound around once in $4r$, the arc BFGC will equal a quarter of a circumference whatever be the height of the cone. For putting

$\pi = 3.14159$, and $a =$ slant height, we shall have

$$\text{BF} = 8\pi r \;\therefore\; \text{BC} = 8\pi r \times \frac{a}{4r} = 2\pi a = \text{ a quarter of the circumference,}$$

rence, a being = radius.

QUESTION XVI. (215.)—*By Mr. George Evans, New-York.*

Through a given point to draw a straight line cutting an ellipse, given in magnitude and position, so that the intercepted part may be of a given length.

FIRST SOLUTION.—By *Analyticus, N. J.*

Let $y^2 + mx^2 = k^2$ (1), be the equation of the given ellipse when referred to its axes; a, b, the coordinates of the given point, when referred to the axes of x, y respectively; $r =$ the distance from (a, b); $\phi =$ the angle made by r with the axis of x; then $x = a - r\cos.\phi$, $y = b - r\sin.\phi$.

* The diagram can be readily supplied.

$\frac{y-y_0}{\sin\varphi} = \frac{x-x_0}{\cos\varphi} = r$, $a^2x^2 + b^2y^2 = a^2b^2$

Substitute the values of x, y in (1), put $k^2-(b+ma^2)=c^2$, b sin. $\phi+ma$ cos. $\phi=v$, sin. $^2\phi+m$ cos. $^2\phi=v'^2$; and there results

$$v'^2r-2vr=2c^2;\ \text{hence, } r=\frac{v\pm\sqrt{(c^2v'^2+v^2)}}{v'^2}\quad - \quad - \quad (2).$$

It appears by (2) that to the same values of v, v'^2, there are in general two values of r, one of which is had by taking the radical, in (2), in plus, the other in minus. Let D = the line to be inscribed; then put the difference of the two values of r=D, and there arises the equation

$$\frac{2\sqrt{(c^2v'^2+v^2)}}{v'^2}=\mathrm{D}\quad - \quad - \quad - \quad - \quad (3):$$

by restoring the values of v, v'^2, and substituting for cos. ϕ its equal $\sqrt{(1-\sin.^2\phi)}$, (3) will be expressed in sin. ϕ and given quantities; whence sin. ϕ being found, the position of the line to be drawn becomes known.

Cor. If $m=1$, the ellipse becomes a circle; then by supposing that the axis of x passes through the given point, a = the distance of the given point from the centre of the circle, and $b=o$; also, $v'^2=1$; then by (3), I have

$$\sin.\phi=\sqrt{\left(k-\frac{\pi^2}{4}\right)}\div a\quad - \quad - \quad - \quad (4);$$

which indicates a well known formula of the problem in this particular case.

Second Solution.—By *Mr. Frederick Furber, Cambridge Col.*

Let the given point be at the vertex and be likewise the origin of rectangular coordinates to which the ellipse is referred; the one of which shall be coincident with the transverse axis, the other with a tangent to its extremity. The equation of the curve is then

$$\mathrm{A}^2y^2+\mathrm{B}^2x^2-2\mathrm{B}^2\mathrm{A}x=0.$$

Changing this into a polar equation and dividing by z, we obtain

$$(\mathrm{A}^2\sin.^2w+\mathrm{B}^2\cos.^2w)z-2\mathrm{B}^2\mathrm{A}\cos.w=0,$$

and if $e=\dfrac{\sqrt{(\mathrm{A}^2-\mathrm{B}^2)}}{\mathrm{A}}$, be the eccentricity, and sin. ^2w be developed; $\therefore$

$$(1-e^2\cos.^2w)z=p\cos.w:$$

hence z, by the question, is given, solving for cos. w, we find

$$\cos.^2we^2z+p\cos.w=z;\quad \cos.^2w+\cos.w\frac{p}{e^2z}=\frac{1}{e^2};$$

$$\therefore\ \cos.w=-\frac{p}{2e^2z}\pm\left(\frac{1}{e^2}+\frac{p^2}{4e^4z^2}\right)^{\frac{1}{2}}.$$

THIRD SOLUTION—*By a Student of Columbia College, N. Y.*

The equation of the ellipse referred to rectangular, co-ordinate axes, being

$$b^2x^2+a^2y^2=a^2b^2 \quad - \quad - \quad - \quad - \quad (1).$$

Calling α and β the co-ordinates of the given point z, the distance of the point from the point of intersection in the perimeter of the ellipse, we have, by analytical geometry,

$$x=\frac{z}{\sqrt{1+A^2}}+\alpha, \quad y=\frac{Az}{\sqrt{1+A^2}}+\beta \quad - \quad (2).$$

Which values substituted in equation (1), we obtain

$$\frac{b^2z^2}{1+A^2}+b^2\alpha^2+\frac{2\alpha b^2z}{\sqrt{1+A^2}}+\frac{a^2A^2z^2}{1+A^2}+a^2\beta^2+\frac{2a^2A\beta z}{\sqrt{1+A^2}}=a^2b^2.$$

This equation, when ordered, &c. becomes

$$z^2+\frac{2\sqrt{1+A^2}(a^2A\beta+\alpha b^2)z}{a^2A^2+b^2}=\frac{(1+A^2)(a^2b^2-\alpha^2b^2-a^2\beta^2)}{a^2A^2+b^2},$$

an equation in z of the second degree, the two roots of which, z' and z'', are found in complete functions of A; the difference of which will also be a function of A, and equal to the given line b. This last equation being resolved, determines the value of A, which is the tangent of the angle that the secant line, passing through the given point, must make with the axis of the abscissas.

QUESTION XVII. (216)—*By Mr. James Macully, Richmond, Virginia.*

Suppose a semicubical and a common parabola to have the same abscissa and semiordinate: it is required to find the difference of their areas; the length of the arch of the semicubical being given, and that of the common a maximum.

FIRST SOLUTION—*By Mr. J. S. Van de Graaff, Lexington, Kentucky.*

Put a=the given arch of the semicubical parabola. The common formula for the arch of a semicubical parabola, when expressed in terms of the co-ordinates only, is,

$$a=\frac{(9x^2+4y^2)^{\frac{3}{2}}-8y^3}{27x^2};$$

from which is obtained the quadratic,

$$x^4+\left(\frac{4y^2}{3}-a^2\right)x^2=16y^3\times\frac{a-y}{27}.$$

We have, also,

$$27ax^2+8y^3=(9x^2+4y^2)^{\frac{3}{2}};$$

and by differentiation, putting

$$v=\frac{18ax-9x(9x^2+4y^2)^{\frac{1}{2}}}{4y(9x^2+4y^2)^{\frac{1}{2}}-8y^2},$$

we obtain, $vdx=dy$.

The arch of the common parabola is to be a maximum, by the question; therefore, putting

$$w=\frac{2x+(4x^2+y^2)^{\frac{1}{2}}}{y},$$

we have $$wy-2x+\frac{y^2}{2x}\times h.l.w=\text{maximum}.$$

Put the differential of this expression $=0$, and substitute for dy its value vdx, there will then result

$$\frac{2xyv-y^2}{2x^2}.h.l.w+\left(\frac{y}{2xw}+1\right)\times\left(\frac{4x+vy}{wy-2x}-wv+2\right)+wv-2=0.$$

This equation, together with the quadratic given above, are sufficient for approximating the values of x and : the required difference of area will then be

$$(\tfrac{2}{3}-\tfrac{3}{5})xy:$$

Second Solution—*By Professor Strong.*

The equations of the semicubical and common parabolas, are

$$px^2=y^3 \text{ - } (1),\quad p'x=y^2 \text{ - } (2), \text{ respectively;}$$

also, if a, s, denote their lengths corresponding to x, y, I have

$$a=\frac{(9y+4p)^{\frac{3}{2}}-8p^{\frac{3}{2}}}{27p^{\frac{1}{2}}} \quad \text{- - - - -} \quad (3);$$

$$s=\frac{(y^2+\frac{1}{4}p'^2)^{\frac{1}{2}}y}{p'}+\frac{p'}{4}\,h.\,l.\,\frac{y+(y^2+\frac{1}{4}p'^2)^{\frac{1}{2}}}{\frac{1}{2}p'} \quad \text{-} \quad (4).$$

By (1) and (2), I have

$$p=\frac{y^3}{x^2},\quad p'=\frac{y^2}{x};$$

these values being substituted in (3) and (4), give, by reduction,

$$27a=\frac{(9x^2+4y^2)^{\frac{3}{2}}-8y^3}{x^2} \quad \text{- - - - - -} \quad (5);$$

$$2s=(4x^2+y^2)^{\frac{1}{2}}+\frac{y^2}{2x}\,h.\,l.\,\frac{2x+(4x^2+y^2)^{\frac{1}{2}}}{y}=\max. \qquad (6).$$

Hence, to the differential of (6) add that of (5) multiplied by an indeterminate; then put the co-efficients of dx, dy, each $=0$, and there will arise two equations which will be reduced to one by eliminating the indeterminate; the equation thus found, and (5), will be sufficient to find x and y; and hence the abscissa and semiordinate of the two curves become known.

Let x', y', denote the common abscissa and ordinate, found as directed; then, $p=\frac{y'^3}{x'^2}$, $p'=\frac{y'^2}{x'}$, are known; also the area of the common parabola when taken $x=0$, and $x=x'$, equals $\frac{2x'y'}{3}$, and the area of the semicubical when taken between the limits $=\frac{3}{5}x'y'$; hence $\frac{2}{3}x'y'-\frac{3}{5}x'y'=\frac{1}{15}x'y'=$ the difference sought becomes known.

Note. I suppose the proposer means, by "semiordinate" in this question, what is usually called the "ordinate."

Third Solution—*By Cartesius, Cincinnati, Ohio.*

The equation of the common parabola is $y^2=px$ - (1),
and that of the semicubical $p'y^2=x^3$ - - - (2);
and since they have the same abscissa and ordinate, they meet at the origin and at another point; for this last, the co-ordinates x and y, in both equations, are equal, and by eliminating y from them, we have $x^2=pp'$ - - - - (3).

The expression for the length of an arc of the semicubical parabola* is

$\frac{8}{27}p'^2\{(1+\frac{9}{4}p'^2x)^{\frac{3}{2}}-1\}=$ a given quantity b, by the question; from which we obtain $p'=\mathrm{F}(x)$. This substituted in equation (3), gives $x^2=p\,.\,\mathrm{F}(x)$; whence $p=\frac{x^2}{\mathrm{F}(x)}$.

This value of p substituted in the following expression for the length of a common parabolic arc,†

$$s=\tfrac{1}{2}x(1+4p^2x^2)^{\frac{1}{2}}+\frac{1}{2p}\,l\,\{2px+\sqrt{(1+4p^2x^2)}\}+c,$$

will be a function of x alone; the differential coefficient of which, being equated to zero, will furnish an equation containing but x

* Ryan's Diff. and Int. Calculus, p. 269.
† Ibid. p. 270.

and known quantities, from which the value of x may be easily found; knowing which y may be found from equation (1). Having thus found x and y, the areas of the two parabolas; and, consequently, their difference will be readily obtained.

QUESTION XVIII. (217.)—*By Mr. W. Nichols, Edinburgh, Scotland.*

Through a given point to draw a straight line cutting two given ellipses having the same centre, so that the distances of this point from the points of intersection shall be in arithmetical progression.

SOLUTION—*By Analyticus, New-Jersey.*

Let $y^2+mx^2=k^2$ - (1); $y'^2+m'x'^2=k'^2$ - (2); be the equations of the ellipse. I shall suppose the ellipse denoted by (2) to fall wholly within that represented by (1). Now, by using the same notation for (1) as in the solution of the XVIth question, and by what was there shown, I have

$$r=\frac{v\pm\sqrt{c^2v'^2+v^2}}{v'^2} \quad - \quad - \quad - \quad - \quad (3);$$

similarly I have for (2)

$$R=\frac{V\pm\sqrt{C^2V'^2+V^2}}{V'^2} \quad - \quad - \quad - \quad (4);$$

R, V, C, V′, respectively denoting the quantities corresponding to r, v, c, v', in (3). Put

$$\frac{v+\sqrt{c^2v'^2+v^2}}{v'^2}=r'' \text{ and } \frac{v-\sqrt{c^2v'^2+v^2}}{v'^2}=r';$$

$$R''=\frac{V+\sqrt{C^2V'^2+V^2}}{V'}, \quad R'=\frac{V-\sqrt{C^2V'^2+V^2}}{V'^2};$$

then by the condition of the question $r''-R''=R''-R'=R'-r'$, which are equivalent to $r''+R'=2R''$ - (5); $R''+r'=2R'$ - (6). Now, a', b', being the co-ordinates of the given point when referred to the axes x', y'; and a, b, when it is referred to x, y; and $\varphi'=$ the angle made by R or r with the axis of x', $\varphi=$ the angle made by R or r with x; then put $\phi-\phi'=\theta=$ the angle made by the axes x, x'. By known formulas I have

$$a'=a\cos.\theta-b\sin.\theta \ (a), \quad b'=a\sin.\theta+b\cos.\theta \ (b);$$

substitute these values of

a', b', in $C^2=k'^2-(b'^2+m'a'^2)$, $V=b'\sin.\phi'+m'a'\cos.\phi'$;

also, put in these expressions for ϕ' its equal $\phi-\theta$; and in $V'^2=$

sin.$^2\varphi'$+m' cos.$^2\varphi'$ put $\varphi-\theta$ for φ'; then v, v'^2, c^2, are expressed in terms of a, b, k^2, m, φ, and V, V'^2, C^2, are expressed in terms of the same quantities, except k^2, m, and of θ, and k'^2, m'; it is hence evident, that by restoring the values of the letters in (5) and (6), and then by substituting for v, v'^2, c^2, . V, V'^2, C^2, their values, there will arise two equations involving a, b, k^2, k'^2, m, m', φ, θ; hence, if all these quantities, except two, are known, the two equations will serve to find the two unknown quantities. If every thing is known, except φ, θ, then φ, θ, can be found; and thence the position of x' and r, with respect of x, are known.

Question XIX. (218.)—*By the same.*

The same data as in the last problem, so that the lines may be in geometrical progression.

Solution.—*By Analyticus, New-Jersey.*

I shall here use the same notation as in the last question; the process is also the same, except that, instead of (5) and (6), I shall have by the conditions,

$$r''\times R'=R''^2 \quad - \quad (1); \qquad R''\times r'=R''^2 \quad - \quad (2);$$

these equations will be expressed in terms of the same quantities as in the last question: whence any two of the quantities can be found when the rest are known, as was there said.

Note. I shall not stop to consider the case when the ellipses are supposed to cut each other, as the process is sufficiently obvious from what has been done.

A Student of Columbia College has ingeniously deduced from his solution to question XVI. the solutions of the last two problems.

Editor.

Question XX. (220.)—*By Calcul, New-York.*

Required the locus of the centres of all the circles that shall touch a given one and also a given ellipse.

First Solution—*By a Student of Columbia College.*

Let the equations of the given ellipse and circle, referred to rectangular co-ordinate axes, with their origin at the centre of the former, be

$$a^2y^2+b^2x^2=a^2b^2 \text{ (A)}, \quad (x'-\alpha)^2+(y'-\beta)^2=r^2 \text{ (B)}.$$

Let x'', y'', be the co-ordinates of any point in the required locus, and r' the radius of the touching circle; then the distance of the centres of the circles $=(r+r')$, the sum of their radii, and we shall have

$$(x''-\alpha)^2+(y''-\beta)^2=(r+r')^2 \text{ (C)}.$$

The distance of this point from the point of contact in the ellipse will be expressed by

$$(x''-x)^2+(y''-y)^2=r'^2 \text{ (D)},$$

which will also be the equation of the tangent circle.

Now, since the ellipse and circle, at the point of contact, will have a common tangent, the expressions for $\frac{dy}{dx}$ from equations (A) and (D) must be equal, we, therefore, have the condition

$$\frac{b^2x}{a^2y}=\frac{x''-x}{y''-y} \text{ (E), and, consequently, } y=\frac{b^2xy''}{x''+(b-1)x};$$

which reduces equation (A) to

$$\frac{a^2b^4x^2y''^2}{(x''+(b^2-1)x)^2}+b^2x^2=a^2b^2,$$

which, resolved by the method of Descartes, gives x, a function of x'', y'', and constants, and thence we obtain y; by means of which r' becomes known by equation (D), which, substituted in equation (C), gives us the equation of the required locus. The curve will have four branches, extending *ad infinitum*; but there will hardly be any need of inserting the long final equation which I have obtained. When $a=b$ the ellipse becomes a circle, and the required locus an hyperbola. There are other methods of solving the question, but the above is the simplest that has occurred to me.

Second Solution—*By Professor Strong, Rutgers College.*

Let $y^2+mx^2=k^2$ - (1) be the equation of the given ellipse; R the radius of the given circle; a, b, the co-ordinates of its centre when referred to the axes of x, y; r the radius of the touching circle; x', y', the co-ordinates of its centre when referred to x, y. Then, because the circles touch each other, I have

$$(x'-a)^2+(y'-b)^2=(\mathrm{R}+r)^2 \quad . \quad (2);$$

$$\text{also, } (x'-x)^2+(y'-y)^2=r^2 \quad . \quad (3);$$

because the circle (rad. r) meets the ellipse. Again, because the ellipse and circle touch each other, they have a common tangent at $(x, y,)$; hence by (1)

$$\frac{dy}{dx}=-\frac{mx}{y}, \text{ and by (3) } \frac{dy}{dx}=-\frac{x'-x}{y'-y};$$

(I have considered x', y', r, as constant in obtaining this equation, because the circle has been supposed to be so placed as to touch the ellipse ;) by comparing the values of $\frac{dy}{dx}$, I have

$$\frac{x'-x}{y'-y}=\frac{mx}{y},\text{ or } y=\frac{mxy'}{x'+(m-1)x} \quad . \quad . \quad . \quad (4).$$

The value of y, as given by (4) when substituted in (1), gives

$$\frac{m^2x^2y'^2}{(x'+(m-1)x)^2}+mx^2=k^2 \quad . \quad . \quad . \quad (5);$$

(5) gives x in terms of x', y', and known quantities; hence y, as expressed by (4), is found in terms of x', y', and known quantities. These values of x, y, when substituted in (3), give r in terms of x', y', and known quantities ; then, by substituting in (2) the value of r thus found, it will give an equation in terms of x', y', and known quantities, which will be the equation of the required locus.

Cor. If $m=1$, (1) becomes the equation of a circle ($k=$ its radius); and by (5)

$$x=\frac{kx'}{\sqrt{x'^2+y'^2}};\text{ then by (4) } y=\frac{ky'}{\sqrt{x'^2+y'^2}};$$

these values of x, y, when substituted in (3), give

$$r=\sqrt{x'^2+y'^2}-k,$$

which substituted in (2) (by supposing that $b=0$) gives

$$2(\mathrm{R}-k)\sqrt{x'^2+y'^2}+(\mathrm{R}-k)^2=a^2-2ax';$$

this can be further simplified by putting $\frac{a}{2}-x'=z$, or $x'=\frac{a}{2}-z$, and there results, after proper reduction,

$$y'^2=\frac{a^2-(\mathrm{R}-k)^2}{(\mathrm{R}-k)^2}\times\left(z^2-\left(\frac{\mathrm{R}-k}{z}\right)^2\right) \quad . \quad . \quad (6).$$

(6) is the equation of an hyperbola, whose semitransverse axis $=$

$$\frac{\mathrm{R}-k}{2},\text{ and } \frac{\sqrt{a^2-(\mathrm{R}-k)^2}}{2}=$$

its semiconjugate ; its foci being at the centres of the given circles, and $a=$ their distance from each other.

Note. The question, considered in this Corollary, is a particular case of the general question ; when it is required to find the locus of all the circles which touch two circles which are given in magnitude and position.

QUESTION XXI. (220.)—*By Wm. Thompson, Saugerties.*

Two simple pendulums are placed near each other and vibrate in the same vertical plane. The vibrating points attract each other according to the law of gravitation. Required their motions, and particularly their small oscillations.

SOLUTION—*By Robert Adrain, LL. D. Professor of Mathematics in the University of Pennsylvania.*

When the general motions of the pendulums are required, we obtain the equations of motion by the general formula,

$$0 = \Sigma\{m\delta x(d^2x - \text{P}) + m\delta y(d^2y - \text{Q})\}.$$

Let a and b be the horizontal distances of the points of suspension; m and m' their masses; r and r' the lengths of the pendulums; x, y, and x', y', the horizontal and vertical co-ordinates of m and m', reckoning from the points of suspension; let D be their distance. Also, suppose the masses m and m' to be between the vertical passing through their points of suspension, and that the point of suspension of m is lower than that of m'. Thus we have

$$\text{D}^2 = (a - (x + x'))^2 + (b + y - y')^2,$$

from which the values of the differential coefficients $\frac{d\text{D}}{dx}$, $\frac{d\text{D}}{dy}$, &c. are easily found.

The forces which affect the motions in x and x' arise only from the mutual gravitation of m and m' in the line D, the moment of which force is $-\frac{mm'}{\text{D}^2}\delta\text{D}$, which, with respect to x, furnishes the moment $-\frac{mm'}{\text{D}2} \cdot \frac{d\text{D}}{dx}\delta x$, which is the value of $m\text{P}\delta x$.

Again, from the equations,

$$x^2 + y^2 - r^2 = 0, \text{ and } x'^2 + y'^2 - r'^2 = 0,$$

we have, by using two multipliers λ and λ', the equations

$$\lambda x\delta x + \lambda y\delta y = 0, \text{ and } \lambda' x'\delta x' - \lambda' y'\delta y' = 0;$$

and the term affecting x is $\lambda x\delta x$, and thus the total coefficient of δx is

$$-\frac{mm'}{\text{D}2} \cdot \frac{d\text{D}}{dx} + \lambda x,$$

and, by writing x' for x, we have a similar coefficient for x'. With regard to the coefficients of δy and $\delta y'$, they are obtained in a similar manner, with the additions of mg and $m'g$, the common gravity being g. And thus, besides the two equations of the circles, we have the four following differential equations, in which dt is constant.

$$md^2x=\lambda x-\frac{mm'}{D^2}\cdot\frac{dD}{dx}$$

$$m'd^2x'=\lambda' x'-\frac{mm'}{D^2}\cdot\frac{dD}{dx'}$$

$$md^2y=\lambda y-\frac{mm'}{D^2}\cdot\frac{dD}{dy}+mg$$

$$m'd^2y'=\lambda y'-\frac{mm'}{D^2}\cdot\frac{dD}{dy'}+m'g.$$

Thus, the problem is reduced to six equations containing the six variables x, y, x', y', λ, λ', besides the term t. It is easy to eliminate λ and λ' and thus reduce the problem to four equations among four variables with t. It is obvious, however, that the problem may be reduced to two differential equations by using two circular arcs, $r\varphi$ and $r'\varphi'$, of which x, y; x', y', are the cosines and sines; and by substituting these cosines and sines the four preceding equations will contain only φ, φ', λ, λ', with dt, and eliminating λ and λ', the problem will be reduced to two equations. We may readily obtain a differential equation of the first order by multiplying by $2dx$, $2dx'$, $2\,dy$, $2dy'$, respectively, and integrating the sum of the products, we have

$$m\cdot\frac{dx^2+dy^2}{dt^2}+m'\cdot\frac{dx'^2+dy'^2}{dt^2}=C+\frac{2mm'}{D}+2g(my+m'y')$$

which is the equation of living forces.

When the motions of m and m' are very small, and the mutual gravitations of the bodies supposed to produce only very small disturbing forces, we consider the small parts of the curves described as straight lines parallel to the horizon, and thus x and x' are reckoned from the lowest points of the circles. Put A for the horizontal distance of the pendulums when vertical, and B the difference of the altitudes of m and m', and $x+x'=y$, and then we have

$$D^2=(A-y)^2+B^2.$$

The accelerative forces arising from common gravity may be denoted by n^2x and n'^2x'; and the disturbing forces are

$$m'\,\frac{A-y}{D^3},\text{ and } m\frac{A-y}{D^3};$$

and, therefore, the equations of motion are

$$1.\qquad 0=\frac{d^2x}{dt^2}+n^2x-\frac{m'(A-y)}{D^3},$$

$$2.\qquad 0=\frac{d^2x'}{dt^2}+n'^2x'-m\frac{(A-y)}{D^3}.$$

Let a and a' be the values of x and y that render $\frac{d^2x}{dt^2}$ & $\frac{d^2x_{\prime}}{dt^2}$ each equal to zero; and, conseqnently, for this position we have the equations

$$0=n^2x-m'\frac{(A-y)}{D^3},\text{ and }0=n'^2x'-m\frac{(A-y)}{D^3}.$$

These values of x and x' determine the position of equilibrium of m and m'; and $\left(\text{putting } C=\frac{m}{n'^2}+\frac{m'}{n^2}\right)$ are found, by means of the equation,

$$A^2D^6=(D^3+C)^2\,.\,(D^2-B^2)$$

This equation has several roots, but in the present case the value of D is very little less than $\sqrt{(A^2+B^2)}$: let this value of D be denoted by D′.

Now, put $x=a+\xi$, $x'=a'+\xi'$; also, put $A-(a+a')=f$,

$$m'\frac{3f^2-D'^2}{D'^5}=\beta,\ m\frac{3f-D'^2}{D'^5}=\beta',\ n^2-\beta=\alpha^2,\ n'^2-\beta'=\alpha'^2,$$

and when we retain only the first powers of ξ and ξ', the preceding differential equations in x and x' are reduced to the two following:

$$3.\quad 0=\frac{d^2\xi}{dt^2}+\alpha^2\xi-\beta\xi',$$

$$4.\quad 0=\frac{d^2\xi'}{dt^2}+\alpha'^2-\beta'\xi.$$

Suppose $\xi'=N\xi$, and let us determine in what cases N can be a constant quantity. By substitution the two last equations become

$$5.\quad 0=\frac{d^2\xi}{dt^2}+(\alpha^2-N\beta)\xi,$$

$$6.\quad 0=\frac{d^2\xi}{dt^2}+\left(\alpha'^2-\frac{\beta'}{N}\right)\xi.$$

Now, it is manifest that if we suppose $\alpha^2-N\beta=\alpha'^2-\frac{\beta}{N}$, the quantity N will be determined by a quadratic equation in terms of the known quantities α, β, α' β': and thus N has two different constant values, N and N′. Let the resulting known values of $\alpha^2-N\beta$, or its equal $\alpha'^2-\frac{\beta}{N}$, be denoted by k and k'; and the last two differential equations become coincident, and each equivalent to either of the equations:

$$7.\quad 0=\frac{d^2\xi}{dt^2}+k\xi,\qquad 8.\quad 0=\frac{d^2\xi}{dt^2}+k'\xi.$$

Each of these equations exhibit a system of coexistent oscillations of m and m', the periodic times of which are respectively

$\frac{2\pi}{\sqrt{k}}$ and $\frac{2\pi}{\sqrt{k'}}$; 2π being the circumference to the radius unity.

The integrals of these equations are

$$\xi = \mathrm{E} \,.\, \sin.(t\sqrt{k}+\varepsilon), \text{ and } \xi = \mathrm{E}' \,.\, \sin.(t\sqrt{k'}+\varepsilon'),$$

in which E, E′, ε, ε', are the arbitrary constants; and the corresponding values of ξ' are

$$\xi' = \mathrm{NE} \,.\, \sin.(t\sqrt{k}+\varepsilon), \quad \xi' = \mathrm{N'E'} \,.\, \sin.(t\sqrt{k'}+\varepsilon').$$

Now, the first two values of ξ and ξ' satisfy equations 3 and 4, and are particular integrals of these two equations containing the two arbitraries E and ε. The second values of ξ and ξ' express another system of coexistent oscillations, and are also particular integrals having the arbitraries E′ and ε'. From these observations we easily infer, that the sums of these partioular values of ξ and ξ', respectively, will also satisfy the same equations, 3, 4, since the terms depending on the coefficients E and E′ do not interfere with each other in making the substitutions for ξ, $d^2\xi$, &c. But these sums contain four arbitrary constants, which are all that can possibly arise in the integration of two differential equations of the second order, involving two variables; and, therefore, the complete values of ξ and ξ' are

$$\xi = \mathrm{E} \,.\, \sin.(t\sqrt{k}+\varepsilon) + \mathrm{E}' \,.\, \sin.(t\sqrt{k'}+\varepsilon'),$$
$$\xi' = \mathrm{NE} \,.\, \sin.(t\sqrt{k}+\varepsilon) + \mathrm{N'E'} \,.\, \sin.(t\sqrt{k'}+\varepsilon').$$

The reality of these equations depends on k and k' being both positive. Since

$$\alpha^2 - \beta\mathrm{N} = k, \text{ and } \alpha'^2 - \frac{\beta'}{\mathrm{N}} = k; \text{ therefore, } \beta\mathrm{N} = \alpha^2 - k\frac{\beta'}{\mathrm{N}} = \alpha'^2 - k;$$

and, therefore, $(\alpha^2-k) \,.\, (\alpha'^2-k) = \beta\beta'$, which always give real positive values of k, when the product $\alpha^2\alpha'^2$ is greater than $\beta\beta'$, as is evident from the equation,

$$k = \tfrac{1}{2}\left\{(\alpha^2+\alpha'^2) \pm \sqrt{((\alpha^2-\alpha'^2)^2+4\beta\beta')}\right\}.$$

QUESTION XXIII. (222.)—*By Mr. Robert Gregory, Nantucket.*

A point is projected upwards along the interior surface of a hollow circular cylinder inclined to the horizon, by an impulse coinciding in direction very nearly with the straight line formed on the lower surface of the cylinder by a vertical plane passing through its axis. Required the motion.

SOLUTION.—*By Professor Strong, Rutger's College.*

Let the motion be defined by the rectangular axes x, y, z; their

origin being at the point of projection, which I suppose to be at some given point on the lower side of the cylinder; the axis of x being the line of common section of a vertical plane through the axis of the cylinder and a horizontal plane passing through the point of projection, and the axis of y is in the horizontal plane at right angles to the axis of x, and the axis of z is vertical to the horizontal plane. Let R = the radius of the cylinder, A = the inclination of its axis to the horizon; then I have

$$\left(2R\sec.A-(z-x\tan.A)\right)\times(z-x\tan.A)-y^2\sec.^2A=0 \quad (1);$$

(1) is the equation of the cylindric surface when referred to the aforesaid axes. Put

$$z-x\tan.A=R\sec.A-r\cos.\theta, \quad y=r\sin.\theta,$$

$$r=\frac{R\sec.A}{\sqrt{\sec.^2A\sin.^2\theta+\cos.^2\theta}} \quad - \quad - \quad (2);$$

these values satisfy (1) as they ought to do: let g = gravity, t = the time from the origin of the motion; then by the formula of Dynamics I have

$$\frac{d^2x\delta x+d^2y\delta y+d^2z\delta z}{dt^2}+g\delta z=0 \quad - \quad - \quad (3).$$

Substitute in (3) the values of δz, δy as given by (2) and we have by putting the coefficient of $\delta x=0$,

$$\frac{d^2x}{dt^2}+\left(\frac{d^2z}{dt^2}+g\right)\tan.A=0 \quad - \quad - \quad (a);$$

$$\frac{d^2y}{dt^2}\times\delta(r\sin.\theta)-\left(\frac{d^2z}{dt^2}+g\right)\times\delta(r\cos.\theta)=0 \quad - \quad (b).$$

By (2) I have

$$\frac{d^2z-d^2x\tan.A}{dt^2}=-\frac{d^2(r\cos.\theta)}{dt^2},$$

this and (a) give

$$\frac{d^2z}{dt^2}+g=\left(g-\frac{d^2(r\cos.\theta)}{dt^2}\right)\times\cos.^2A,$$

and by (2) $$\frac{d^2y}{dt^2}=\frac{d^2(r\sin.\theta)}{dt^2};$$

by substituting these values in (b), I have

$$\frac{d^2(r\sin.\theta)\times\delta(r\sin.\theta)+d^2(r\cos.\theta\times\delta(r\cos.\theta)\cos.^2A}{dt^2}$$
$$-g\cos.^2A\delta(r\cos.\theta=0 \quad - \quad - \quad (c);$$

it evident that in (c) I may change δ into d, which being done, by integration, etc. I have

$$\frac{\left(d(r\sin.\theta)\right)^2+\left(d(r\cos.\theta)\right)^2\times\cos.^2A}{dt^2}$$

$$-2g\cos.^2Ar\cos.\theta=c=\text{const.} \quad (d).$$

Again, r as given by (2) can be changed to

$$r=\frac{R}{\sqrt{\sin.^2\theta+a\cos.^2\theta}}\ (a=\cos.^2A);\ \text{hence}$$

$$r\sin.\theta=\frac{R}{\sqrt{1+a\cot.^2\theta}},\quad r\cos.\theta=\frac{R}{\sqrt{\tan.^2\theta+a}};$$

put $\tan.\theta=s$; then these values being substituted in (d) give by reduction

$\dfrac{aR^2\frac{ds^2}{dt^2}}{(a+s^2)^2}-\dfrac{2gaR}{\sqrt{a+s^2}}=c$; but at the origin of the motion $s=0$, and

if the initial value of $\dfrac{ds}{dt}=v$, then $c=\dfrac{R^2v^2}{a}-\dfrac{2Rga^{\frac{3}{2}}}{a}$; hence,

$$\frac{aR\frac{ds^2}{dt^2}}{(a+s^2)^2}+\frac{2g(\sqrt{a^2+as^2}-a)}{\sqrt{a+s^2}}=\frac{Rv^2}{a} \quad (e);$$

the integral of (e) will give sin. a function of t, whence s becomes known at any time, and hence x, y, z are easily found (by what has been done) and the position of the particle is determined at any given time. (e) is easily adapted to the case proposed in the question when the impulse coincides very nearly in direction with the lower side of the cylinder; for on this hypothesis, v is to be considered as a very small quantity; hence supposing s to be always of the same order as v, I may put θ for s; then by reducing (e) on the supposition that quantities of a higher order than v^2 are to be neglected; I have (by putting $\dfrac{g\sqrt{a}}{R}=n^2$)

$dt=\dfrac{d\theta}{\sqrt{v^2-n^2\theta^2}}=\dfrac{d\left(\frac{n\theta}{v}\right)}{n\sqrt{1-\left(\frac{n\theta}{v}\right)^2}}$; by integration $e+nt=\text{arc}\left(\sin.=\dfrac{n\theta}{v}\right)$; hence, $\theta=\dfrac{v}{n}\sin.(nt+e)$, ($e$=const.); but at the origin of the motion when $t=0$, $\theta=0$; hence, $e=0$; $\therefore\ \theta=\dfrac{v}{n}\sin.nt$ (f); whence the position of the particle is easily found at any time.

NOTE. That $v=\frac{V}{R \sec. A}$; in which V is that part of the initial velocity of projection which is perpendicular to the vertical plane passing through the axis of the cylinder.

ACKNOWLEDGMENTS, &c.

The following Gentlemen favoured the Editor with Solutions to Questions in Art. XXV. No. XI. Vol. II. The figures annexed to the names refer to the questions answered by each, as numbered in that article.

Professor Strong, Rutgers College, New-Brunswick, New-Jersey; Mr. Marcus Catlin, Elizabethtown, and Analyticus, New-Jersey, solved all the Questions.

A Student at Rutgers College, N. B., N. J.; Mr. Frederick Furber, Cambridge College; and a Student of Columbia College, New-York, solved all but 21, 22; Messrs. James and Gerardus B. Docharty, Long Island, solved all but 20, 21, 22; Omicron, North Carolina; Messrs. Silas Warner, and Benjamin Wiggins, solved all but the last five questions; Mr. John M. Wilt, solved 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12; Tyro, Brooklyn, solved 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12; Mr. O. Root, solved 2, 4, 5, 8, 10, 13, 15, 18; Mr. C. Gill, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19; Mr. N. Vernon, 6, 8, 13, 14; Mr. J. S. Van de Graaff, 15, 17; Robert Adrain, LL. D. Professor of Mathematics in the University of Pennsylvania, solved 1, 22; Mr. Lenhart, 14; Messrs. Townsend, William Hewlett, and Jacob S. Underhill, 1, 2, 3, 6; Cartesius, Cincinnati, Ohio, solved 17; Messrs. Patrick Carlin, and Thomas Mooney, 2, 5, 6, 11, 12; Mr. John Carmody, 2, 5; and Arithmeticus, New-York, 2. The solutions of Messrs. Carlin, C. Gill, John Carmody, and Mooney, were received too late for publication.

N.B. Those gentlemen who have no acknowledgments for solutions to their own questions, have not sent any solution.

ARTICLE XXVII.

Periodical Account of New and Interesting Publications.

ANNALES DE MATHEMATIQUES,* pures et appliqueès, par M. GERGONNE; Tom. xx. Nos. 7, 8, 9, 10, 11, 12; Tom. xxi. Nos. 1 and 2: for January, February, March, April, May, June, July, and August, 1830.

No. 7, for January, 1830.

The first part of this number contains an investigation of the number of points through which any one of the varieties of a curve, or curved surface of any degree, can be made to pass. It is well known that if m be the degree of the equation of a plane curve, it can generally be made to pass through the number of points denoted by $\frac{m+1}{1} \cdot \frac{m+2}{2} - 1$, and that this is the limit of the number; and if of a curved surface, it can be made to pass through the number of points denoted by $\frac{m+1}{1} \cdot \frac{m+2}{2} \cdot \frac{m+3}{3} - 1$. But in every degree curves and curved surfaces have many varieties; and the number of points thro' which these varieties can be made to pass varies greatly: thus, though a line of the second order can be made to pass through five points given upon a plane; yet, if the line be a circle, we can only make it pass through three points. Again, though a surface of the second degree can be made to pass through nine points given in space; yet, if that surface be a sphere, it can only be made to pass through four points.

In the second part M. Gergonne calls the attention of the reader to the method given by Newton for the approximation of incommensurable roots of numerical equations, the most convenient and expeditious method that we have, but one that does not always ensure success; as the cases in which it is sure, and when the contrary, have not yet been sufficiently examined. When this method is successful, it gives values always greater or less than the true one; continually approximating to it without being able to assign the limit of the error by which each coefficient is affected, a thing not permitted in the present state of analysis. Newton's method requires a complement, without which it cannot really be employed; and it is from geometry that M. Gergonne determines this complement. M. G. concludes by observing, that one of the great advantages of Newton's method is, that it applies equally well to equations having any number of unknown quantities.

* Annals of Mathematics, pure and applied.

Nos. 8 & 9, February and March, 1830.

Since the discovery of the differential calculus, the explanation of its fundamental principles has been changed and varied in many different ways. M. Gergonne, in the numbers before us, presents, in the form of an essay, a new exposition of these principles, which, to him, seems to excel in simplicity and clearness all others. He first remarks, that it is from elementary analysis, and its applications to the theory of curves, that we are led to consider the derivatives of polynomials as functions of one or more variables: so that the art of determining the derivatives of these polynomials should be familiar to every well taught student, before even that he is made acquainted with the existence of a branch of analysis which has for its special object the determining of these derivatives. He thus founds the whole differential calculus upon this principle. For instance, let there be a polynomial of a finite or unlimited number of terms, in each of which the variable x is affected by a coefficient as well as by any number of constant exponents; if we have another polynomial of the same number of terms, each of which shall be deduced from its corresponding one in the former; multiplying this by the exponent of x, and dividing this exponent by unity, the new polynomial will be termed the *derived function* of the first; which, on the other hand, will be the *primitive* function.

No. 10, April, 1830.

The first article of this No. consists of a demonstration of the principle of *virtual velocities*, in the case of machines, by M. Bobilier, director in chief of the school of arts and sciences at Angers. In his demonstration he substitutes, for any given machine, another, whose conditions of equilibrium it is easy to establish, and which, with the exception of an infinitely small displacement, shall be of a nature to produce the same effect as the first. His choice consists of a series of wheels, placed upon a common axis, which, in fact, is the same with the demonstration given by Carnot in his *Fundamental Principles of Equilibrium and of Motion.*

In the second article, an anonymous writer complains, with some reason, of the species of affectation which induces writers, on the elements of geometry, to give intricate and difficult demonstrations of what might be made simple and easy. He takes, as an example, the well known expression for the volume of a truncated pyramid or cone, right or oblique, which has parallel bases, which some writers have proved, by considering the cone as the difference of two pyramids or of two cones.

In the third article, M. Valles supplies what is wanting in a demonstration of the parallelogram of forces given by Mr. King, two years since, in the *Transactions of the Cambridge Philosophical Society*, which was first discovered to be incomplete by M. Gergonne.

In the fourth article, M. Lenthèric slightly modifies the process by which we arrive at the superior limit of the roots of equations, so as to render them easier of access to young beginners.

In the following article, an anonymous correspondent demonstrates, in a very simple manner, "that there is no whole number which has

not among its multiples either a number expressed by unity, followed by one or more zeros; a number expressed by one or more 9s; or a number expressed by one or more 9s, followed by several zeros."

Finally, in the last article, there are demonstrations of five long theorems in analytical geometry.

No. 11, May, 1830.

The number for May is devoted exclusively to a memoir of M. Gergonne, upon *maxima* and *minima*, in functions of one or more variables. M. G. recalls the attention of the reader to his previous *exposé* of the principles of the differential calculus, viz.—1st, we obtain by them the definition of a new operation in calculus; 2dly, the practical rules of this operation upon all known functions, rigorously deduced from its definition; and 3dly, certain general symbols, proper to indicate this operation, executed one or more times upon functions of one or more variables, whose form we suppose entirely indeterminate.

M. Gergonne first considers an explicit function fx of the single variable x. We are accustomed to say, that the value of x will be that which suits the maximum of fx, if, in representing by i a very small quantity, we have $f(x\pm i)<fx$; but this method of enunciation is not correct, and our author takes great care to avoid it. For a very small value of i, we might have $f(x\pm i)<fx$, and still not have the value which answers to the maximum value of fx. In order that it may be so, it is besides necessary that the inequality hold good for all values of i, from i to 0; this is neglected by the greater part of elementary writers. After having established the known rules for the determination of maxima and minima of explicit functions of one variable, and having shown the distinction between maxima and minima, M. Gergonne conducts us in an extremely simple manner to what he says respecting functions of several variables. Let $S=f(x,y)$ be a function of two variables x and y, entirely independent of each other; let us conceive two entirely arbitrary equations established between these variables and a third variable s; then s can be considered as being uniquely a function of these last; whence it follows, that the condition, common to their maximum and minimum values, will be expressed by the equation $\frac{dS}{ds}=0$, and that there will be either one or the other, according as for a value of s drawn from this equation, the function $\frac{d^2S}{ds^2}$ is *negative* or *positive*. Or, again, if we would represent by the characteristic δ the differential or *variation* relative to s, we shall have $\frac{dS}{ds}=\frac{dS}{dx}\delta x+\frac{dS}{dy}\delta y$; but because the supposed relations between x, y, and s, are arbitrary, δx and δy should remain independent, so that $\frac{dS}{ds}$ could not be equal to zero—inasmuch as we shall have separately $\frac{dS}{dx}=0$, $\frac{dS}{dy}=0$, and such will consequently be the equations which will give the only system of values of x and y which could answer to the

maxima and minima values of the function S. By means of these equations we shall obtain simply

$$\frac{d^2S}{ds^2}=\frac{d^2S}{dx^2}\delta x^2+\frac{2d^2S}{dxdy}\delta x\delta y+\frac{d^2S}{dy^2}\delta y^2;$$

and that this function may preserve the same sign continually, whatever be the values of δx, δy, as it should do, for either max. or min. it will be necessary that we have

$$\left(\frac{d^2S}{ds^2}\right)^2-\frac{d^2S}{dx^2}\cdot\frac{d^2S}{dy^2}<0 \quad (A)$$

a condition omitted by Euler and supplied by Lagrange, who has exceeded the necessary limits as much as Euler fell short of them. If, in fact, we had $(A)=0$, the result would merely be that $\frac{d^2S}{ds^2}$ is of the form $M(P\delta x+Q\delta y)^2$, and that, consequently, whatever values be attributed to $\delta x, \delta y$, $\frac{d^2S}{ds^2}$ will never have a sign different from that of M; but if we took δx, δy, in such a manner as to have $P\delta x+Q\delta y=0$; as, in general by virtue of this hypothesis, $\frac{d^3S}{ds^3}$ will not become identically nothing; it then follows, that there will be, generally speaking, neither a max. nor a min. which is conformable to known theories. But if $\frac{d^3S}{ds^3}$ contain the function $P\delta x+Q\delta y$, and if $\frac{d^4S}{ds^4}$ preserves invariably the same sign with M, whatever be δx, δy, there will then be a maximum or minimum value. M. Gergonne proceeds to treat with the same logical rigour, and in the same form, of maxima and minima, of explicit functions of more than two independent variables. He thence passes to the case in which these variables are connected by more or less numerous relations, and terminates by investigating maxima and minima of implicit functions of one or more variables: so that his memoir, besides being of a sufficiently limited and concise form, may be regarded as a complete treatise upon the subject.

No. 12, June, 1830.

In the first article of this number M. Gergonne demonstrates, after the manner of M. Crelle, "that every given number is always the divisor of another number, expressed by periods of given numbers, followed by a certain number of zeros." For instance, having the period 4813 given, there is no given number which has not a multiple of the form 4813,4813.....481348130000.....0000.

In the second article M. R. S. occupies himself with some algebraic expressions extremely complicated and little known.

In the third article M. Le Barbier gives a formula for determining, with many decimal places, the logarithms of large numbers when the tables of these logarithms are very limited. The explanation of the ingenious methods to which the author has recourse would require too much space to be included within the narrow limits of this journal.

In another article M. Lentheric proves "that the product of the three sides of a right-angled triangle" is always exactly divisible by 60.

Finally, in the last article, M. Pagliani resolves this problem: "It is required to find 1000 consecutive numbers of the natural series, such that the sum of their cubes shall be itself a cube." The author finds the first of these to be 1134, and, consequently the last is 2133; and that the sum of their cubes is 16830. He observes, that we should come to the same result by taking for the first of these numbers —499 and for the last +500; the sum of their cubes will then be the cube of 500. M. P. thinks that no other solution can be deduced, except by a very complicated analysis.

Vol. XXI. Nos. 1 & 2, July and August, 1830.

At the end of his exposition of the principles of the differential calculus, which we have previously noticed, M. Gergonne pledged himself to give various applications of it; a part of which pledge he fulfilled in his treatise on maxima and minima. The number that we announce is entirely devoted to the author's application of the same principles to the theory of plane curves. He treats successively of the transformations of co-ordinates, of points of inflexion, of the various species of multiple points, of normals, of centres and radii of curvature, of *devellopeès* and osculating curves of different orders; in fine, he makes applications of the general formula that he has obtained for curves comprised in the most general equation of the second degree with two indeterminates. He thinks fit, in conclusion, to give certain general conditions, which every final equation, obtained by the aid of his notation, should satisfy, and which form so many characters proper to recognise the errors which may have glided into calculations. It is a point according to him, too much neglected by other writers. He founds his whole theory upon the law of homogeneity; which, though misconceived by some writers, he esteems one of the fundamental laws of science.

In the second number M. Valles demonstrates four theorems: the first of which is this—"If a, b, c, are the lengths of the right lines which join the summits of a rectilineal triangle included in the centre of a circle, r will be given by the equation of the third degree, which has no second member.

$$\left(\frac{1}{r^3}\right)-\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{1}{r}\right)-\frac{2}{abc}=0."$$

Σ.

Annals of philosophy, conducted by Richard Taylor, F.S. A.L.S. &c. and Richard Phillips, F.R.S.L. &c. Vol. 8. Nos. 43, 44, 45, 46, 47, 48, for July, August, September, October, November, and December, 1830.—New series.

Of this work, which has been issued for many years past, there have been 2 series: the 1st of which was entitled the "Annals of Philosophy," and the second or new one, "The Philosophical Magazine and Annals of Philosophy"—a title with which we should have prefaced our article. It has been and is devoted to researches in all the branches of mathematical and physical science, although a far greater portion

of it is taken up with the latter than the former; of which we shall take a rapid and imperfect survey.

No. 43, for *July*.—The first article of this No. is devoted to an account of the *Measurement* (by Trigonometry) *of the principal Hills* of Swaledale, Yorkshire—by John Nixon, Esq.

The second article consists of a paper upon the *Obliquity* of the *Ecliptic*—by William Galbraith, Esq. M. A., who is considered one of the most distinguished mathematicians in Great Britain. Mr. Galbraith states, that sufficient attention has not been bestowed on some parts of this subject, particularly on that arising from a small error in the latitude, which, as will presently appear, produces a double effect on the difference between the summer and winter obliquity; by this means rendering that small error very sensible.

Let ω be the summer obliquity, ω' the winter, λ the latitude, z the zenith distance in summer, z' that in winter, and ϵ the error in latitude; then it will readily appear, that in

$$\text{Summer} \quad - \quad - \quad \omega = \lambda + \epsilon - z \quad - \quad - \quad (1)$$

$$\text{Winter} \quad - \quad - \quad \omega' = z' - (\lambda + \epsilon) = z' - \lambda - \epsilon \quad - \quad - \quad (2)$$

$$\text{Whence} \quad - \quad \omega - \omega' = 2\lambda - (z + z') + 2\epsilon = \left\{ 2\lambda - (z + z') + 2\epsilon \right\} \quad (3)$$

But $2\lambda - (z + z')$ should be $= 0$, if the summer and winter obliquities are the same, as it is probable they are; whence

$$\omega - \omega' = \Delta\omega = 2\epsilon \quad - \quad - \quad - \quad - \quad - \quad (4)$$

Or the difference of the summer or winter obliquity is affected by the sum of the errors in the latitude arising from using an erroneous table of refractions, a constant error in the instrument, &c.

The sixth article consists of " A direct method of finding the shortest distance between two points on the earth's surface, when their geographical position is given"—by James Ivory,* Esq. M.A. F.R.S. &c. Mr. Ivory here explains a solution of this problem different from any hitherto obtained; according to which the shortest distance between two points on the earth's surface is expressed by means of their latitudes and the inclination of the equator to the great circle of the celestial sphere that passes through them. The rest of the No. is entirely devoted to other than mathematical subjects.

No. 44. *August*.—In the article marked 15 of the volume, at page 114 of this number, Mr. Ivory illustrates the series obtained by him in article 6, by applying to it an example given by M. Puissant, at page 42 of the additions to the *Connaissance des Tems*, for 1832, relative to the distance between two points on the earth's surface.

The next article in No. 16, consists of a paper by Prof. Gauss on a new principle in mechanics, which we shall enunciate as follows:

" The motion of a system of material points, connected together in any manner whatsoever, whose motions are modified by any external restraints whatsoever, proceeds in every instance in the greatest possible accordance with free motion, or under the least possible constraint; the measure of the constraint which the whole system suffers in every particle of time, being considered equal to the sum of the products, of

* Mr. Ivory is well known to our readers as being ranked the first mathematician in Great Britain.

the square of the deviation of every point, from its free motion into its mass."

The next article is a continuation of the trigonometric measurements before spoken of.

No. 45. *September.*—The only mathematical articles in this number are, "An attempt to explain theoretically the different refrangibilities of the rays of light according to the hypothesis of undulations"—by the Rev. J. Challis, Fellow of Trinity College, Cambridge, and of the Cam. Phil. Soc.; and the conclusion of the paper on trigonometric measurements.

No. 46. *October.*—The first article of this number, which is Art. 38 of vol. 8, consists of "Further observations on the obliquity of the ecliptic"—by Mr. Galbraith, of Edinburgh; in which he institutes comparisons, &c. between the results that Bradley, Maskelyne, Piazzi, and many others, have obtained for the obliquity of the ecliptic.

Art. 42 contains an interesting paper "On the solid of least attraction"—by Samuel Sharpe, Esq. which we regret not being able to insert on account of the narrow limits of the Diary.

In article 44 we find Prof. Bessel's "Additions to the theory of eclipses and the methods of calculating their results."

No. 47. *November.*—The first article of this number, which is Art. 48 of the volume, consists of a "Memoir on the calculus of variations"—by Hugh Ker Cankrien, Esq. M.A. Trin. College, Cambridge. The object proposed is to investigate the relation which the variables involved in a proposed function must have to one another, in order that a definite value of the function shall be a maximum or minimum. The most common form in which this function is proposed, is the integral taken between the limits of an expression containing the variables and the differential coefficients of one of them considered as a function of the other. The author proceeds to the solution of the easier problems, such as the brachystochronon; he then shows how far this solution is applicable to the more difficult—and finally, in what way the solution of these may be accomplished.

Article 52 consists of a continuation of the Theory of Eclipses, by Professor Bessel.

No. 48, for *December*, in Art. 64, we have the conclusion of Prof. Bessel's additions to the Theory of Eclipses.

Art. 67 consists of a paper on determining the Longitude by occultations of the fixed stars—by Mr. Squire. Σ.

Mecanique Celeste of Laplace, translated with a commentary. By Nathaniel Bowditch, LL.D. Fellow of the Royal Societies of London, Edinburgh, and Dublin, &c. &c. Vol. I. Boston. Hilliard, Gray, Little & Wilkins, 1829. 4to. pp. 746.

This translation of the most enduring monument of the fame of Laplace, reflects the highest honour upon our country, as well for the talents and profound learning evinced by our distinguished countryman (upon whom the genius of our literature should look as one of her most

highly gifted sons) as for the typographical execution, which is truly splendid, and as we think, the best specimen of typography that has ever issued from the American press. The American translation will by no means suffer when compared with the original for elegance and perspicuity; and the learned commentator has supplied all the steps that are wanting, has amply illustrated the results of the investigations, and has in fact untied the gordian knot, the secret of which has hitherto been disclosed but to the chosen few, whom nature and the study of years have qualified to enter into the *penetralia* of the analytical investigations of this most sublime but difficult author.

N. B. For a more elaborate view of the work of Laplace and the Translation of Dr. Bowditch, the reader is referred to the American Quarterly Review, No. X. for June, 1829, at page 310, where Professor Renwick, of Columbia College, is the author of an excellent article upon Celestial Mechanics, and upon the Astronomy of Laplace in particular, and to the same, No. XIV. for June, 1830, where the first article consists of a review of Dr. B.'s work, together with an interesting account, by the same gentleman, of the four last books of the Mecanique Celeste.

Σ.

ASTRONOMIE PRATIQUE—USAGE ET COMPOSITION DE LA CONNAISSANCE DES TEMS. By L. B. FRANCOEUR, Professor of the Faculty of Sciences of Paris, of the College of Charlemagne, etc. etc. Paris. Bachelier. 1 vol. 8vo. pp. 472. 1830.

This new work of M. Francoeur, who is already distinguished as the author of many and valuable original productions, amply fills up a blank that has long existed in astronomical science, viz. the want of some publication which should have for its special object practical astronomy, i. e. one which offered the most extensive application of astronomical formulæ and the ephemerides to the problems of a science whose long and complicated calculations ever present innumerable difficulties. This want of some work by means of which we may be speedily initiated into astronomical calculations, is supplied by the work before us, which offers in a single volume of a convenient and portable form, a collection of all the questions liable to occur in practical astronomy, together with examples of all cases of calculation presented under forms which render their execution extremely easy. It is divided into three parts: the first of which contains an explanation of the signification and use of all the expressions, &c. employed in the *Connaissance des Tems*, a French periodical almanac too well known to need any comment, together with the means of verifying their exactness.—Each question is accompanied by an example of the mode of calculation. The second part comprises the theory of the principal problems in astronomy, to be resolved by the navigator or observer of the heavens. We here find the indication of all the methods of observations that can be used in determining the latitude and longitude of a place and azimuth. The second part terminates with a complete exposition of the theory of the tides. The third part treats of the composition and uses of astronomical tables; it contains the process which conducts to the determination of the obliquity of the ecliptic, the preces-

sion of the equinoxes, and the corrections pending from the phenomena of nutation and aberration. The work is accompanied by seventeen tables, whose construction and uses are explained by the author. Σ.

Memoire sur les equations generales de l'equilibre—et du nouvement des corps solides élastiques et des fluides lu à l'Academie roy des science, le 12 Octobre, 1829. Par M. Poisson.

Memoir upon the general equations of the equilibrium, and of the motion of solid elastic bodies and of fluids, read by M. Poisson to the Royal Academy of Sciences, Oct. 12, 1829.

This is the third Memoir of M. Poisson upon the same subject, and forms a conspicuous part of the 20th number of the Journal of the Polytechnic School. One of his previous papers, inserted in the additions to the *Conn. des Tems*, for 1829, gave rise to the long dispute between the author and Mr. Ivory relative to the theory of the figure of the planets, contained in the Mecanique Celeste, a contest which ceased without an acknowledgment on the part of either, though M. Poisson is judged to have amply proved the incorrectness of Mr. Ivory's arguments, which may be seen in the Philosophical Magazine, vols. 69, 70, 71, 72, for the years 1827, 1828. Σ.

Theorie Analytique du systeme du Monde—par G. de Pontecoulant, ancien éleve de l'Ecole Polytechnique. Paris. Bachelier, 1829. 2 vols. 8vo. pp. 508, 504.

This work is highly spoken of by all the foreign reviews,* &c. as one of much genius and erudition, and its young and talented author is said to be fast treading in the footsteps of Laplace and Lagrange. He has divided it into five books, of which two occupy the first volume and the remaining three the second. The first book contains an investigation of the general laws of equilibrium and of motion, and is divided into seven chapters. The second treats of the motion of revolution of the celestial bodies, and is divided into ten chapters. The third contains the theory of comets, and is divided into four chapters. The fourth treats of the rotatory motion of the celestial bodies, and is divided into five chapters. The fifth contains an investigation of the figure of the celestial bodies, and is divided into six chapters. We have here given but an outline of the work, whose reasoning, like that of Laplacc, is purely analytical, and into the sounder part of which we have not as yet had time to penetrate, but upon the merits of which we hope to be able to pronounce at some future period. Σ.

* Foreign Quarterly Review, for November, 1829; Revue Encyclopedique, for June, 1830.

ARTICLE XXVIII.

NEW QUESTIONS

TO BE RESOLVED BY CORRESPONDENTS IN NO. XIII.

QUESTION I. (222.)—*By Mr. William Vogdes, Philadelphia.*

Given $\left\{ \begin{array}{c} x-y+\sqrt{\left(\dfrac{x-y}{x+y}\right)}=\dfrac{20}{x+y} \\ x^2+y^2=34 \end{array} \right\}$ to determine the values of x and y by a quadratic.

QUESTION II. (223.)—*By Analyticus, New-York.*

A father dying, bequeaths an estate of ten thousand dollars in the following manner: in case that his wife (who was pregnant at the time of his death) should have a daughter, the wife was to have $\frac{2}{3}$ and the daughter $\frac{1}{3}$; but in case of a son, the wife was to have $\frac{1}{3}$ and the son $\frac{2}{3}$. Now, it so happened that the wife brought forth two sons and one daughter. Required, the respective shares of the wife, sons, and daughter.

QUESTION III. (224.)—*By Mr. Wm. Lenhart, New-York.*

The fractions $\frac{2}{3}$ and $\frac{1}{3}$ are such that their difference is equal to the sum of their cubes: are there other fractions having the same properties?

QUESTION IV. (225.)—*By the same.*

Given $\left\{ \begin{array}{l} (x+y)(xy+1)=18xy \\ (x^2+y^2)(x^2y^2+1)=208x^2y^2 \end{array} \right.$ to deteamine x and y.

QUESTION V. (226.)—*By the same.*

Given $axy=x^2-by^2$, and $x^3y^3=x^2+by^2$, to determine x and y, in terms of a and b by simple equations; and find general expressions for a and b which shall render x and y rational.

QUESTION VI. (227.)—*By Mr. O. Root, Vernon, N. Y.*

Given, the base and sum of the sides of a plane triangle to construct it when the angles at the base are as 1 to 2.

QUESTION VII. (228.)—*By Mr. Benjamin Wiggins.*

In a given parabola, it is required to determine the greatest inscribed ellipse, one of whose axes shall be parallel to the base of the parabola.

QUESTION VIII. (229.)—*By Mr. James Macully, Richmond, Va.*

In a given ellipse, it is required to inscribe the greatest possible equilateral triangle.

QUESTION IX. (230.)—*By Mr. Francis Sherry, New-York.*

It is required to inscribe the greatest trapezoid in a given segment of a circle.

QUESTION X. (231.)—*By the same.*

Given $\left(\frac{\sin.{}^2\varphi}{\cos.{}^2\varphi}\right)^{\sec.{}^2\varphi} = \frac{4\cos.{}^2\varphi}{3}(\tan.{}^2\varphi)^{\tan.{}^2\varphi}$

to determine φ.

QUESTION XI. (232.)—*By Cartesius, Cincinnati, Ohio.*

It is required, to divide a quadrant of a circle into three arcs which shall be to each other in the ratio of three given right lines.

QUESTION XII. (233.)—*By Mr. William Lenhart.*

In a triangle, having given the base, the ratio of the sides and one of the angles adjacent to the base, double the vertical angle; to determine the triangle.

QUESTION XIII. (234.)—*By Mr. E. Giddens, Lockport, N.Y.*

The diameters of two circles are 2 and 5 respectively, and the distance between their centres is 10; to describe another circle touching the two and their tangent.

QUESTION XIV. (235.)—*By Cartesius, Cincinnati, Ohio.*

Required, the locus of the intersections of the ordinates of an ellipse, with the perpendiculars let fall from its centre upon the tangents at the extremities of their ordinates.

QUESTION XV. (236.)—*By the same.*

It is required, to find the caustic by reflection, the reflecting curve being the locus of the intersections of a tangent to a given circle, with a line perpendicular to it, passing through a given

point in the circle and the focus of incident rays being at the pole of the reflecting curve.

QUESTION XVI. (237.)—*By Mr. Gerardus B. Docharty.*

Required, the content of the least sphere that shall touch three paraboloids given in magnitude and position.

QUESTION XVII. (238)—*By a Student of Columbia College, N. Y.*

It is required to investigate the nature and properties of the curve along which a circle must roll, in order that its centre shall generate a cycloid.

QUESTION XVIII. (239.)—*By Mr. Marcus Catlin, Elizabethtown, N. J.*

In a given cone it is required to determine the greatest inscribed cone having its vertex in a given point in the slant surface.

QUESTION XIX. (240.)—*By the same.*

Required the locus of the centres of the bases of all the cones inscribed in a given cone having their vertices in a given point in the slant surface.

QUESTION XX. (241.)—*By Professor Thomson, Nashville University, Tennessee.*

Required, the expression for the solidity of a solid of revolution, the equation of whose curve is $y=(\text{tang}.x)^2$, radius unity, the coordinates being taken on the concave side of the curve, and their origin at the centre of the generating circle.

QUESTION XXI. (242.)—*By Mr. C. Gill, New-York.*

A and B are given points in an indefinite straight line XY; from B draw any right line BT, and describe a circle through A, having its centre in XY, which shall touch BT in T. To find the equation of the curve which is the locus of T; determine its area and length, and the surface and solidity of the solid generated by its revolution round XY.

QUESTION XXII. (243.)—*By Mr. J. Leslie Payne, Halifax, Nova Scotia.*

The upper extremity of a simple pendulum of given length is moved, according to a given law, along a given curve. Required the motions of the pendulum compatible with this restriction.

Acknowledgment to Contributors.

Several questions, interesting and *difficult*, were received, but unaccompanied with their solutions, generally under a fictitious signature: such contributions shall never be noticed in the Diary. The Editor received, also, several questions, accompanied with their solutions, but they were too late for insertion in the present number.

MISCELLANIES.—*Prize subject proposed by the Society of Natural Sciences of Liege, for* 1831.—A gold medal of the value of 50 florins (low country currency) will be awarded to the author of the most complete disquisition on the chemical history of the colouring matter of the blood, accompanied by an investigation of the uses to which this substance may be applied in the arts. The memoirs, written either in French, German, or Latin, must be sent before the 15th July, 1831, to M. Wellekeus, secretary-general of the society.—*Correspondence Mathe. et Phys.* 2d series, vol. 3, p. 159, 1830.

Prize subject proposed by the Academy of Dijon.—A prize of 300 francs ($60) will be awarded to the inventor of the best theory accounting most satisfactorily for the changes of temperature, both great and small, which accompany chemical action, in the following manner: 1. By determining if the changes of temperature which take place during chemical action (by this is meant both synthesis and analysis) depend upon a single cause; or, are the result of many causes in any case which may occur. 2. By assigning to each of these causes (should there be more than one) its proper share of influence. 3. By clearly explaining, supporting by experiments, and, if needful, by calculations which shall tend to give them a high degree of probability, the new hypotheses which may arise. The memoirs must be sent to the President of the Academy, before the 1st of May, 1831.—*Bulletin des Sciences Mathématiques*, for April, 1830.

Prize subjects proposed by the College of Pharmacy of Paris.—A prize of 1500 francs ($300) will be awarded for the best dissertation upon the following questions: 1. It is required to indicate in a precise manner the conditions which determine the transformation of alcohol into acetic acid. 2. To indicate the phenomena accompanying this transformation, and the products which result from it.

Also, a prize of 1000 francs ($200) will be awarded for the best memoir upon the following question: It is required to indicate a series of characteristics proper to distinguish the vegetable alkalies, either from each other or from other organic substances, which shall also be sufficiently certain to be employed in cases of legal medicine. The memoirs must be addressed, before the 1st of June, 1831, to M. Robiquet, at the College of Pharmacy.—*Bulletin vide supra.*

Pure Selenium.—It is announced that perfectly pure Selenium may be obtained, at the rate of 4 gold frederics (about $18) the Cologne ounce (29 grains), by addressing a franked letter, with the remittance, to the Ducal Counting-house (*Comptoir*) at the mines of Harzgrove, in the Duchy of Anhalt.—*Bulletin vide supra.*

EVENING AMUSEMENTS,

OR

THE BEAUTIES OF THE HEAVENS

DISPLAYED;

IN WHICH

SEVERAL REMARKABLE APPEARANCES

TO BE OBSERVED

ON VARIOUS NIGHTS IN THE HEAVENS, DURING THE FIRST SIX MONTHS OF THE YEAR

1831,

ARE DESCRIBED.

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ARTICLE XXIX.

EVENING AMUSEMENTS,

OR

THE BEAUTIES OF THE HEAVENS DISPLAYED.

JANUARY.

The moon's latitude on the 1st., at noon, is fifty-six minutes south, in the twenty-second degree of the fifth sign, ♌; it decreases to the 2d when she passes the ecliptic in her ascending node, at a little past seven, in the morning, in the third degree of the sixth sign, ♍. Her northern latitude increases to the 9th, when it is at seven in the evening, five degrees, eight minutes, thirty-two seconds, in the third degree of the ninth sign, ♐; and it then decreases to the 16th, when she passes the ecliptic, in her descending node, at six in the evening, in the first degree of the twelfth sign, ♓; and her southern latitude now increases, being on the 23d, at noon, five degrees, twenty-eight minutes in the twelfth degree of the third sign, ♊; and it then decreases to the 29th, when she passes the ecliptic, in her ascending node, about noon, in the sixth degree of the sixth sign, ♍.

The moon passes under Regulus, in Leo, about ten in the evening, on the second of this month, elevated about sixteen degrees above the eastern point of the horizon.

On the 5th, the moon completes her last quarter, at fifty-eight minutes past five, in the evening, in the constellation of Virgo;—distant about six degrees from Spica Virginis.

On the 13th is new moon, at forty-one minutes past eight in the evening, but without an eclipse, as she now is about three and a quarter degrees north of the ecliptic. The enlightened hemisphere of the moon is now turned from us, and consequently, that face of the moon which is always turned towards the earth, receives none of the sun's rays at the time of new moon. The moon, however, is faintly illuminated by the reflected light of our earth, which appears to a spectator in the moon thirteen times larger than she does to us; and at the same time, the whole of the illuminated portion of the earth is turned towards her. The moon, at her reappearance, on the 15th, is seen in the middle of Capricornus, having Venus fifteen degrees to the west, and Mercury between Venus and the moon, about four degrees from the latter.

On the 19th, the moon is in the twelfth degree of the first sign, at seven in the evening, about forty-one degrees elevated above the S. W. point of the horizon, and thirteen degrees to the west of Mars, both to the west of the meridian, the moon being on the meridian at sunset.

Distances of the moon's centre from the principal stars, east of her, at four minutes past seven in the evening.

The distange from Spica on the 3d, is 30° 39′; from Antares, on the 5th, 52° 29′; from Arietes, on the 18th 38° 43′; from Aldebaran, on the 21st, 26° 59′ from Pollux, on the 23d, 43° 26′; from Regulus, on the 26th, 35° 38; from Spica, on the 29th, 48° 25′.

Distances of the moon's centre from the principal stars, west of her, at four minutes past seven in the evening.

The distance from Pollux, on the 3d, is 60° 10′; from Regulus, on the 5th, 48° 4′; from Spica, on the 8th, 30°27′; from (α) Pegasi, on the 20th, 41° 50′; from Arietes, on the 24th, 50° 10′; from Aldebaran, on the 26th, 44° 30′; and from Pollux, on the 29th, 42° 36.

Position of the heavens on the 19th, at seven o'clock in the evening.

In this position of the heavens, that part of the ecliptic which is comprehended between twenty-five degrees of ♒, and twenty-five degrees of ♌, is above the horizon. The constellations of the zodiac, which correspond with this part of the ecliptic, are, a part of Aquarius, Pisces, Aries, Taurus, Gemini, Cancer, and a part of Leo.

The beautiful and brilliant constellation Orion, is situated between the ESE. and SE. Betelgueze is elevated thirty-two degrees, and forms an equilateral triangle with Procyon in the Little Dog, and Sirius in Canis Major, the former is almost due east thirteen degrees, and the latter seven degrees above the horizon.

Rigel in the left foot, is 27 degrees elevated, and towards the SE. it is in the same direction with the Pleiades, and Hyades in Taurus, the former is approaching the meridian. Aldebaran in Taurus, is elevated fifty-one degrees, and about half the distance between it and the horizon, the three stars, constituting the belt of Orion, are seen, and about eight degrees above the belt, Bellatrix, in the left shoulder.

EbS. the bright foot of the Twins, (γ) second magnitude, thirty-two degrees high. ENE. The two heads of the twins, Castor and Pollux, second magnitude, Castor, (α) elevated thirty degrees, and Pollux, (β) about five degrees lower.

Towards the same point of the horizon you will observe Capella, a very remarkable star of the first magnitude, elevated sixty degrees, and (β) of the second, about seven degrees lower. SSE. Menkar, in the mouth of the whale, (α) second magnitude, fifty degrees high. This constellation is now in the most favourable position to be observed, as it is partly on the meridian. But, though it occupies a vast space in the heavens, it contains but five stars of any note. Above the Pleiades, and approaching the zenith of New-York, Algol, in Medusa's head, (β) second magnitude.

The constellation Ursa Major, occupies that portion of the heavens to the east of the meridian, between the NE. and the N. This is one of the most remarkable constellations in the northern hemisphere, because it is composed of seven very conspicuous stars.

The two stars (β) and Dubhe, in the body of the Great Bear, are called *guards*, or *pointers*, because a line passing through them points to the north pole; and the two stars β and γ in the body of Ursa Minor, are called the guards or pointers of the Little Bear:—The former is nearly on our meridian, and fifteen degrees below the north pole.

Nearly in the direction of the pointers of the Great Bear, and about five times the apparent distance between them, reckoning from Dubhe, is Alruccabah, or north pole star, in the tail of the constellation Ursa Minor.

In the western part of the heavens, just past the meridian, and nearly in our zenith, is Alamak, in the left foot of Andromeda, (γ) second magnitude: below it, and SW. is Mirach, (β) second magnitude, in her girdle, seventy-three degrees above the horizon; and in the same line in her head (α) second magnitude, fifty-eight degrees. This last star forms a great square with three other stars, also of the second magnitude, which we perceive nearer the horizon; they belong to the constellation of Pegasus; the lowest is Markab, (α) in his wing; the eastern point of the square is Algenib, (γ) also in his wing, and the western point is Scheat, (β) in his thigh. Algenib in Pegasus and (α) Andromeda are nearly in the equinoctial colure; the distance between them is about fourteen degrees, and the distance of the former from the equinoctial point Aries, is nearly the same.

A little to the south of the west, the group composing the constellation of the Dolphin may be easily perceived, elevated thirteen degrees. Turning more to the north, Altair, in the Eagle, (α) first magnitude, is nearly setting. Between WbN. and NW. the Swan is remarkable by its five principal stars disposed in the form of a cross, and is still in a good disposition to be observed. The first star, Arided, is the most elevated, thirty-one degrees; the second, Albiero, in the beak, is the lowest, ten degrees. Above the Swan, but more to the north, the constellation of Cassiopeia, cannot fail of being known by its principal stars, which form a sort of a W. Scheder, (α) third magnitude, is the principal star in this constellation, elevated sixty-nine degrees.

NW. the Harp claims our notice; Lyra or Wega, of the first, elevated ten degrees, and β a quadruple star of the third magnitude, are the principal stars in this constellation. Nearly between Schedar and Lyra, Alderamin in Cepheus, may be easily perceived forty-four degrees high. NWbN. the two bright stars in the head of the Dragon, are about fourteen degrees above the horizon.

Of the Planets.

Mercury is an evening star, and at his greatest elongation on the 11th; consequently, he will be about fourteen degrees above the horizon at sunset; they, therefore, who have a clear aspect nearly towards the SW. will have many good opportunities of observing him during several nights. Mercury will be in his inferior conjunction on the 26th.

Venus is an evening star, and on the 19th, she will appear about nine degrees above the horizon at sunset; and therefore she will not set till nearly three quarters of an hour after the sun, and a few minutes before Mercury.

Mars is an evening star, whose latitude on the 19th is forty-five minutes north in the twenty-seventh degree of the first sign, ♈, and he will be in his quadrature on the 14th.

Jupiter is an evening star, and is about ten degrees above the horizon at sunset; he will be in conjunction on the 20th, and till that time an evening star.

Saturn is an evening star, his latitude on the 1st, is 1° 31′ N. in the second degree of the sixth sign; he rises about nine o'clock, a few minutes after Regulus [α] of the first magnitude in the constellation Leo.

Uranus is in conjunction on the 31st, and till that time an evening star. His latitude on the first is thirty-seven minutes S. in the ninth degree of the eleventh sign, ♒; the beginning of this month is distinguished by six planets being evening stars at the same time, that is, Mercury, Venus, Mars, Jupiter, Saturn, and Uranus.

FEBRUARY.

The moon's latitude on the 1st, at noon, is three degrees, eight minutes, north, in the eighth degree of the seventh sign, ♎, it increases to the fifth, when it is at seven in the evening, 5° 16′ nearly, in the thirtieth degree of the eighth sign, Scorpio; and it then decreases to the 12th, when she is in conjunction with the sun in longitude 10 signs, 23° 18¾′ moon's latitude 42¾ minutes N. at 4h. 58¼ m. afternoon, mean time at London, or 2¼ m. P. M. mean time at New-York. The sun will be centrally eclipsed on the meridian at half past twelve, N. Y., in longitude 81° 28¾′ West, and in latitude 35° 5¼′ North.

The eclipse will be first seen at sunrise, in longitude 125° 18′ W. at lat 14° 10′ N. The eclipse will be last seen at sunset in long. 36° 22′ West, and lat. 40° N. The duration of the entire eclipse for the whole earth is 5h. and 3m. nearly.

The sun will very nearly be centrally eclipsed at the following places: California, several of the provinces of Mexico, Texas, Opelousas, and St. Helena, in Louisiana, Monticello, in Mississippi, Cahawba, in Alabama, Athens in Georgia, Abbeville and Laurens, in South Carolina, Hillsborough and Chapel Hill, North Carolina, Richmond, Yorktown, Petersburg, Williamsburg, in Virginia, the SE. counties of Maryland and Delaware, near the eastern shores of New-Jersey, Long Island, Rhode Island, and Massachusetts, in the Atlantic Ocean, Halifax, in Nova Scotia, in the Gulph of St. Lawrence, and a part of the island of Newfoundland.

The sun will be nearly 11½ digits eclipsed on the south limb at the following places, towns, and cities:—Several of the internal provinces of Mexico, Alexandria, Louisiana; Natchez, Mississippi; Tuscaloosa, Alabama; Warm Springs, North Carolina; Monticello, the seat of the immortal Jefferson, and Fredericksburg, Virginia; Annapolis, Mary-

land; Philadelphia, Pennsylvania; New-York; New-Haven, Connecticut; Providence, Rhode Island; and Salem, Massachusetts.

The places where it will be eclipsed eleven digits, are a few degrees north of the above places; and for a less number of digits, the number of degrees north of the above path increase. In a similar manner, the number of degrees south of the central path, when eleven, nine, six, &c. digits are eclipsed, may be traced out on a terrestrial globe or a map of the world.

The moon will pass the ecliptic at a few minutes past noon, on the 12th, in his descending node, at which time we will have new moon, her latitude increases till the 19th, when it is about 5° 21′ in the second degree of the third sign, ♊, at six in the afternoon; and then decreases to the 26th, when she passes the ecliptic in her ascending node, then she is full moon at a few minutes before noon on that day:—the moon eclipsed, invisible in the United States, visible in Asia, and only the latter part visible in Europe.

Distances of the moon's centre from the principal stars, east of her, at four minutes past seven in the evening.

The distance from Antares, on the 3d, is 33° 21′; from Aldebaran, on the 17th, 30° 7′; from Pollux, on the 20th, 33° 12′; from Regulus, on the 23d, 26° 16′; from Spica, on the 27th, 27° 42′; and from Antares, on the 28th, 61° 10′.

Distances of the moon's centre from the principal stars west of her, at four minutes past seven in the evening.

The distance of the moon's centre from Regulus, on the 1st, is 43° 47′; from Spica, on the 4th, 26° 41′; from Arietes, on the 19th, 33° 49′; from Aldebaran, on the 22d, 40° 8′; and from Pollux, on the 25th, 37° 53′.

Position of the heavens on the 19th, at eight o'clock in the evening.

In this position of the heavens, that part of the ecliptic which is comprehended between the beginning of the signs ♈ and ♎ is above the horizon. The constellations of the Zodiac which correspond to this part of the ecliptic, are Pisces, Aries, Taurus, Gemini, Cancer, and Leo, in which constellations the following stars may be observed:

In the western part of the heavens, the sight is attracted by the splendour of the beautiful constellation Orion, which is in a very favourable position to be examined. Betelgueze (α) in his right shoulder, first magnitude, has passed the meridian, elevated fifty-five degrees. Bellatrix, (γ) in his left shoulder, second magnitude, is SSE. fifty-two degrees high; and in the same direction below Bellatrix, Rigel, (β) in the left foot, first magnitude, is elevated thirty-two degrees. The three stars in the belt, (δ, ϵ, ζ,) second magnitude, are easily known by their being disposed in the same line: between the belt and Rigel we see three other stars, which form almost a straight line with the foot of Canis Major. These three stars represent the sword of Orion; the highest, third magnitude, is the hilt of the sword; the next, fourth magni-

tude, is remarkable by its nebulous appearance, which gives it a resemblance to a comet; it is in the middle of the sword. The third, 3d magnitude, in the right knee, represents the point of the sword. Between Betelgueze and Bellatrix, the eleventh, fourth magnitude, is a little higher up, and makes with them a triangle; it is in the head of Orion. The small stars which are seen between Bellatrix and Aldebaran, represents the lion's skin, which Orion is holding in his left hand. Finally, the stars which form his right arm and his club, occupy a place between Betelgueze, the foot of the Twins, and the southern horn of Taurus.

EbS. Regulus is elevated thirty-two degrees; EbN. Denebola is elevated fifteen degrees; both these stars are in the constellation Leo: the former is of the first magnitude, and the latter of the second.

A line passing from Dubhe through (γ) in the opposite angle of the trapezium, which forms the body of the Great Bear, will nearly intersect Cor Caroli, an extra constellated star of the second magnitude in the neck of Chara, whose distance from the latter star is nearly double the distance between the former two. Between Cor Caroli, Denebola and Vindiematrix, in Virgo, is the constellation Coma Berenices, or Berenice's Hair, so named from its resemblance to hair; these three stars form an isosceles triangle. The other principal stars above the horizon have been described in the month of January.

Of the Planets.

During this month, three planets are visible every morning, at the same time, Mercury, Jupiter, and Uranus. Mercury is a morning star during the whole of this month; his latitude is 3° 30′ N. on the 1st, in 30° of the tenth sign; it decreases till the 21st, when he passes the ecliptic. The greatest elongation takes place on the 20th, at which time he can be easily seen above the eastern part of the horizon before sunrise. Mercury becomes stationary on the 2d, and also on the 24th.

Venus is an evening star during this month, but scarcely visible on account of her proximity to the sun; her latitude on the 19th, is 1° 25′ in the fifteenth degree of the twelfth sign, ♓. Venus is stationary on the 14th.

Mars is an evening star at the distance of two signs, fifteen degrees from the sun on the 19th.

Jupiter is a morning star, but invisible till towards the latter part of the month; because he is too near the sun.

Saturn is an evening star, and in opposition on the 17th; his latitude on the 19th, is 1° 39′ N. in the twenty-ninth degree of the fifth sign.

MARCH.

The moon's latitude and longitude at four minutes past seven in the evening.

On the first, the latitude is 3° 55′ N. in the nineteenth degree of the seventh sign; it increases till the 5th, when the latitude is 5° 14′, in the seventh of the ninth sign; the latitude then decreases till the 12th, when she passes the ecliptic in her descending node; her latitude then increases till it is 5° 13′ S. in the third degree of the third sign; her

latitude continues to decrease till the 25th, when she passes the ecliptic in her ascending node in the eighth degree of the sixth sign.

The moon completes her third quarter on the 6th, at fifteen minutes past noon. On the 14th is new moon, at fifty-three minutes past two in the morning; she completes her first quarter twenty minutes past five in the afternoon, and we shall have full moon on the 28th, at twenty-five minutes past three in the morning.

Distances of the moon's centre from stars east of her, at four minutes past seven in the evening.

The distance from Antares on the 2d, is 37° 22′; from (α) Aquilæ, on the 4th, 65° 8′; from Fomalhaut, on the 6th, 75° 46′; from Pollux, on the 19th, 36° 17′; from Regulus on the 22d, 29° 38′; from Spica on the 26th, 31° 37′; from Antares, on the 29th, 41° 10′; and from (α) in the Eagle, on the 31st, 68° 32′.

Distances of the moon's centre from stars west of her, at four minutes past seven in the evening.

The distance from Regulus on the first, is 51° 34′; from Spica on the 3d, 22° 44′; from Antares on the 8th, 36° 42′; from Aldebaran on the 20th, 23° 5′; from Pollux, on the 24th, 34° 17′; from Regulus on the 28th, 47° 22′; and from Spica on the 30th, 19°.

The sun enters the equinoctial point Aries on the 21st of this month, at four minutes past three in the morning.

Position of the heavens at nine o'clock on the evening of the twentieth.

In this position one half of the ecliptic from the third degree of ♉, to 3° of ♍, is above the horizon, which it cuts in 17° from W. to N. and 17° from E. to S. The constellations of the zodiac which correspond with it, are, Aries, Taurus, Gemini, Cancer, Leo, and Virgo; the principal stars of which are:—

WNW. a little to the north, Arietis of the second, and (β) of the third magnitude, are now very near the horizon, and will soon cease to be discernible.

WbN. the Pleiades are elevated twenty-two degrees, Menkar is setting in the W. and between Menkar and Aldebaran, very near the latter, the Hyades can be observed; Aldebaran is elevated twenty-six degrees higher up, the northern horn of Taurus, (β) of the second magnitude, is elevated forty-four degrees WSW. the Twins are in a good position to be observed with advantage. The two heads, the former of the first and the latter of the second magnitude, are five degrees from each other. Castor is elevated about seventy degrees; Pollux is a little more to the east, and a few degrees lower. Under Castor, almost in a line with the first stars of Orion, we cannot fail to recognise the bright foot of Pollux, (γ) second magnitude elevated fifty degrees above the SSE. point of the horizon. This constellation is also advantageously placed for observation, and its principal stars can be easily traced out. ESE. Denebola, of the second magnitude, is elevated forty-seven degrees. The bright star, Spica, in the Virgin, is elevated ten degrees, and in the same direction as Denebola.

EbS. you observe Vindiematrix, of the third magnitude, elevated thirty degrees. EbN. Arcturus, in the skirts of Bootes, of the first magnitude, is elevated twenty-three degrees; and above Arcturus, Cor Caroli, second magnitude, elevated forty-eight degrees.

ENE. the Northern Crown may now be observed, and will easily be known by its stars being disposed in the form of a circle. Alphecca, or the Gem, second magnitude, is about twelve degrees high.

Of the Planets.

Mercury is a morning star; his latitude on the 1st is 1° 12′ S. in the fifteenth degree of the eleventh sign; on the 31st, his latitude is 1° 38′ S. in the fifth degree of the first sign Aries.

Venus is an evening star; her latitude on the 1st, is 1° 15′ S. in the 28th degree of the twelfth sign; at sunset, she is above the horizon, and she sets about seven in the evening. Her latitude on the 25th, is twenty-seven minutes S. in the twenty-seventh degree of the seventh sign. At sunset, she is about twenty degrees above the west point of the horizon; consequently, she does not set till nearly one hour and three quarters after the sun; she is stationary on the 27th.

Mars is an evening star; he is seen between SW. and SWbS.; at sunset he is elevated about sixty-two degrees on the 1st, in the twenty-first degree of the second sign, to the west of Aldebaran.

Jupiter is a morning star; his latitude on the 1st is 28′ S. in the ninth degree of the eleventh sign; he rises an hour and twenty minutes before the sun, at the ESE. point of the horizon; and on the 25th, he rises two hours and a quarter before the sun.

Saturn on the 1st, is nearly in the same point when referred to the heavens, as Regulus in Leo, his latitude 1° 40′ N. and the latitude of Regulus 27′; and their longitudes are nearly the same. This planet will be visible during the greatest part of each night in this month.

Uranus is a morning star, and appears in the same part of the heavens with Jupiter; the difference of their longitudes is only three degrees, and their latitudes differ but ten minutes.

APRIL.

The moon finishes her third quarter on the 5th, at seven minutes past seven in the morning; we shall have new moon on the 12th, at four minutes past eleven in the morning; she finishes her first quarter on the 19th, at thirty-one minutes past one in the afternoon; and full moon is on the 25th, at thirty-three minutes past seven in the evening. The moon is then between (β) in the constellation Libra, and Spica in the Virgin.

Distances of the moon's centre from the principal stars east of her, at four minutes past ten in the evening.

The distance from (α) in the Eagle is on the 1st, 57° 3′; from Fomalhaut, on the 4th, 55° 57′; from Regulus, on the 18th, 31° 9′; from Spica, on the 22d, 33° 12′; from Antares, on the 25th, 42° 58′; from (α) in the Eagle, on the 28th, 59° 58′; and from Fomalhaut, on the 30th 70° 1′.

Distances of the moon's centre from the principal stars west of her, at four minutes past ten in the evening.

From Spica, on the 1st, 43° 36′; Antares, on the 4th, 34°; Aldebaran, on the 18th, 49°; Pollux, on the 21st 45° 18′; Regulus, on the 25th, 57° 51′; and Spica, on the 27th, 28° 37′.

Position of the heavens at nine o'clock in the evening, on the nineteenth of this month.

In this position, the constellations of the zodiac above the horizon are: a great part of Taurus, Gemini, Cancer, Leo, Virgo, Libra, and a part of Scorpio, the principal stars of which are: between WNW. and WbN. Aldebaran in Taurus, is eight degrees above the horizon. Towards the west, Castor and Pollux, are fifty degrees elevated, and in this direction you will observe the constellation Orion approaching the horizon; Bellatrix is ten degrees, and Betelgueze sixteen degrees above the horizon. SSW. Regulus in Leo is sixty degrees above the horizon, and SSE. Denebola has nearly the same elevation. SE. Spica Virginius is elevated thirty degrees, and nearly in the same direction Vindiematrix fifty degrees.

Between SEbE. and ESE. (α) and (β) in the balance are elevated about twelve degrees. The other principal stars east of the meridian, are:—Arcturus, EbS. forty-four degrees high; Mirach, in the same constellation, E. 42°; and Cor Caroli, due east, 68°. NE. Lyra is elevated ten degrees. The guards or pointers of the Great Bear are on the meridian; nearly in the direction of the pointers, and about five times the apparent distance between them, reckoning from Dubhe, is Alruccaba, the north pole star in Ursa Minor.

The principal stars west of the meridian, are:—Cor Hydra, SSW. elevated thirty-eight degrees; SWbW. Syrius elevated ten degrees, and Procyon, WSW. thirty-three degrees; and NWbW. Capella, thirty-two degrees above the horizon.

Of the Planets.

Mercury is a morning star till the 6th, when she is in her superior conjunction at nineteen minutes past two in the morning; and she is an evening star during the remainder of the month.

Venus is an evening star; her latitude on the 19th, is forty-two minutes N. in twenty-eight degrees of the second sign, ♉.

Mars is an evening star; his latitude on the 1st, is 1° 17′ N. in 11° of the third sign; he is little more elevated than Aldebaran in Taurus.

Jupiter is a morning star; his latitude on the 1st, is 33′ S. in the sixteenth degree of the eleventh sign; he is stationary on the 28th.

Saturn is an evening star; his latitude on the 1st, is 1° 39′ N. in the twenty-sixth degree of the fifth sign; he is nearly on the meridian at sunset, and very near Regulus in Leo.

Uranus is a morning star; his latiude on the 1st, is 38′ S. in 14° of the eleventh sign:—Uranus and Jupiter are very nearly in the same apparent part of the heavens.

MAY.

The moon commences her last quarter on the 4th, at thirty-nine minutes past ten o'clock in the evening; we shall have new moon on the 11th, at five minutes past seven in the evening; first quarter at sixteen minutes past eleven in the morning, on the 18th; and full moon on the 26th, at four minutes past eleven in the morning.

Distances of the moon's centre from the principal stars east of her, at four minutes past ten in the evening of the following day.

On the 2d, from Fomalhaut 48° 15′; on the 3d, from (α) Pegasi 51°, 48′; on the 15th, from Regulus 35° 28′; on the 18th, from Spica, 18° 46′; on the 23d, from Antares, 34° 17′; on the 25th, from (α) in the Eagle, 62° 34′; on the 30th, from (α) Pegasi 54° 41.′

The moon's distance from stars west of her, at four minutes past ten in the evening.

On the 1st, from Antares, 30° 35′; on the 7th, from (α) Aquila, 60° 20′; on the 18th, from Pollux, 42° 13′; on the 21st from Regulus, 42° 50′; on the 24th, from Spica, 25° 33′; on the 29th, from Antares, 39° 10.

Position of the heavens at half past nine in the evening of April 20.

In this position, the constellations of the zodiac above the horizon, are, Gemini, Cancer, Leo, Virgo, Libra, and Scorpio, the principal stars of which are: WNW. Castor and Pollux, twenty-seven degrees elevated; W. Asellus Boreus, in Cancer, 34°; WSW. Regulus is elevated forty-two degrees; and SW. Denebola is sixty degrees above the horizon.

Vindiematrix, Cor Caroli, and Alioth in the Great Bear, are on the meridian, and Spica is approaching it elevated forty-one degrees SE. you will observe Cor Scorpio elevated eight degrees.; and between Spica and Cor Scorpio, α and β, the principal stars in the balance, can be readily observed.

EbS. Alpheca in the Northern Crown, is elevated fifty-five degrees: nearly E. are Ras Algethi and Ras Alhagus, the former in Hercules, elevated thirty degrees, and the latter in the Serpent Bearer, twenty-five degrees.

ENE. nearly, Lyra is elevated thirty degrees, and in the same direction you can see Albireo in the beak of the Swan, fourteen degrees high.

EbN. the Eagle is rising, and SE. Arcturus is elevated 65 degrees.

The principal stars to the west of the meridian, except those already mentioned, are, SWbW. Cor Hydra, elevated 20°; W. Procyon 10°; NWbN. Capella is elevated 11°; and between Capella and the Zenith, the four stars forming the trapezium of the Great Bear are in a favourable position.

Of the Planets.

Mercury is an evening star till his inferior conjunction on the 26th, he will be at his greatest elongation on the 3d, between the Hyades and Pleiades in Taurus, and stationary on the 14th.

Venus is an evening star; the latitude on the 1st, is 1° 14′ N. and longitude two signs 11° 40′; her distance between Venus and Mercury is about ten degrees, the latter planet will set before the former.

Mars is an evening star; his latitude on the 1st, is 1° 19′ N. and longitude two signs 28° 45′; Mars will not set till four hours after sunset.

Jupiter is a morning star; he will be in quadrature, or three signs distant from the sun on the 12th, at thirty-four minutes past six in the in the morning.

Saturn is an evening star; he will be three signs distant from the sun, at thirty-four minutes past six in the afternoon.

Uranus is an evening star; he will be in quadrature on the 5th a few minutes before noon; he will be stationary on the 18th.

The beginning of this month is very favourable for observing all the planets in the course of one night; four of them are evening stars, and two morning stars.

JUNE.

The moon enters her last quarter on the 3d, at twenty-four minutes past ten in the morning. New moon is on the 10th, at fifty-five minutes past one in the morning; she completes her first quarter on the 16th at three minutes past eleven in the evening; and full moon is on the 25th, at four minutes past two in the morning. The moon is then near the beginning of the sign Capricornus.

Distances of the moon's centre from stars east of her, on the following days, at four minutes past ten in the evening.

On the 2d, from Arietis, 60° 28′; on the 15th, from Spica, 39° 48′; on the 19th, from Antares, 37° 10′; on the 21st, from (α) Aquila, 65° 5′; on the 24th, from Fomalhaut; 64° 24′; and on the 27th, from (α) Pegasi, 46° 27′.

Distances of the moon's centre from stars west of her, on the following days at four minutes past ten in the evening.

On the 1st, from Antares, 75° 22′; on the 4th, from (α) in the Eagle, 68° 14′; on the 5th, from Fomalhaut, 46° 45′; on the 17th, from Regulus, 39° 43′; on the 21st, from Spica, 34° 24′; and on the 26th, from Antares, 48° 9′.

Position of the heavens at ten o'clock in the evening, about the middle of the month.

Ursa Minor is on the meridian, and Alruccabah, the polar star may be easily discovered at the end of his tail.

Corona Borealis is also on the meridian, about ten degrees south of the zenith of New-York, and the head of the Serpent about ten degrees lower. Between the polar star and the NNE. point of the horizon, is the constellation of Cassiopeia, the principal stars of which are five in number, appearing like an inverted chair, Schedar is about fifteen degrees high. Alederam in Cepheus, is towards the NE. elevated forty degrees. Scheat in Pegasus is about six degrees high. A line drawn from this star to the head of Cepheus will pass through Lacerta.

Deneb, or Arided in the Swan, is about forty degrees high, in NEbE. The stars in the head of Draco, are nearly in the same direction sixty-eight degrees high; due E. about twenty degrees high, is Delphinus; forty-three degrees elevated, Albireo in the beak of Cygnus; and EbN. fifty-eight degrees high, Vega in Lyra. Atair in Aquila, is thirty-degrees high, EbS. nearly Antinous is more to the south, and the three stars in his foot are about twenty-six degrees high. The achronical rising of the Eagle is about the beginning of this month. Ras Algethi in Hercules is about fifty-six degrees high, and the whole of the constellation extends nearly to the zenith. Ras Alhagus in Serpentarius, is fifty-two degrees high, both constellations in the direction of the SE. point of the horizon. Serpentarius, called also Ophiuchus, with serpents, was known to the ancients, and its extent noticed by the poets.

In 1604, a star appeared in the eastern foot of Ophiuchus; but is said to have disappeared in 1605.

SbE. Antares in Scorpio, is twenty degrees high below the foot of Serpentarius.

Antares forms a large triangle with Arcturus and Spica Virginis westward, or with Vega eastward.

Castor in Gemini, and Capella in Auriga, are both setting nearly due west, Cor Leonis or Regulus, is thirteen degrees high. Cor Caroli is in the same direction, sixty degrees elevated. WbS. Denebola in Leo, is elevated 35 degrees.

Virgo occupies the space from WSW. to SSW. Vindiematrix is about forty-five degrees high; and Spica SWbS. thirty degrees high. SW. Arcturus in Bootes, is sixty-two degrees elevated.

Of the Planets.

Mercury is a morning star on the 1st, in latitude, 3° 17′ S. longitude two signs and one degree, nearly; he will be stationary on the 8th, and at his greatest elongation on the 20th, and may be seen in the morning between the Pleiades and Aldebaran in Taurus.

Venus is an evening star; her latitude on the 1st is 2° 3′ in the nineteenth degree of the fourth sign; she will appear a few degrees south of Castor and Pollux.

Mars is an evening star, and nearly in the same latitude and longitude with Venus; they are both in the constellation of the Twins. Jupiter is a morning star; he is stationary on the 11th, and will be nearly on the meridian at sunrise, about the middle of this month.

Saturn is an evening star, and nearly in the same point of the heavens is Regulus in Leo, about the middle of June.

Uranus is a morning star, and about nine degrees to the west of Jupiter; they are both nearly in the plane of the ecliptic.

Erratum.—Page 192, in last line of Question XIV. for "their ordinates," read *those* ordinates.

THE

MATHEMATICAL DIARY;

CONTAINING

COLLECTIONS OF ORIGINAL QUESTIONS,

PROPOSED AND RESOLVED

BY INGENIOUS CORRESPONDENTS.

TO WHICH ARE ADDED

Important and Valuable Communications

UPON VARIOUS BRANCHES

OF ABSTRACT SCIENCE,

A MEMOIR OF LAGRANGE, WITH A PORTRAIT; MATHEMATICAL DIALOGUES, TOGETHER WITH AN INDEX OF AMERICAN MATHEMATICIANS.

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ADVERTISEMENT.

NUMBER XIII. of the Mathematical Diary completes the second volume. This periodical, begun in January, 1825, was conducted by Robert Adrain, LL.D., Professor of Mathematics and Natural Philosophy in Columbia College, New-York, till 1826, when he was appointed Professor in Rutgers College, New-Brunswick, New-Jersey; he was subsequently elected Professor in the University of Pennsylvania, Philadelphia, where he now resides.

In consequence of Dr. Adrain's change of residence, I became the editor and publisher of the Diary. Neither labour nor expense has been spared to make it useful and interesting to the public, and especially to those scientific gentlemen, who contributed to its support.

In the list of the contributors, are enrolled the names of the most eminent mathematicians in the United States; it is sufficient to mention Dr. Bowditch, Dr. Adrain, Professor Nulty, Dr. Anderson, Professor Strong, and Professor O'Shannessy. The list of all the contributors occupy the five last pages of the present number.

The two volumes of the Diary now published, contain two hundred and forty-six new Questions with their Solutions, in all the branches of Mathematical and Physical Science.

The present number contains several original papers on various subjects, and also a Portrait* of J. L. Lagrange, with a biographical sketch of that *great analyst.*

Each succeeding number of the Diary will contain not only new Questions and Solutions, but also original communications on Philosophical and Mathematical subjects, together with a Memoir of some celebrated mathematician. J. RYAN.

New-York, March 22d, 1832.

* This has been presented to the Mathematical Diary by Mr. Samuel Ward, 3d, a young gentleman of great promise, as will be seen from his contributions to this and the preceding number of the Diary.

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MATHEMATICAL DIARY.

NUMBER XIII.

ARTICLE XXX.

SOLUTIONS

TO THE QUESTIONS PROPOSED IN ARTICLE XXVIII. NO. XII.

QUESTION I. (222.)—*By Mr. William Vogdes, Philadelphia.*

Given $\left\{ \begin{array}{c} x-y+\sqrt{\left(\dfrac{x-y}{x+y}\right)}=\dfrac{20}{x+y} \\ x^2+y^2=34 \end{array} \right\}$ to determine the values of x and y by a quadratic.

FIRST SOLUTION—*By Mr. W. Hanson, Lawrenceville, New-Jersey.*

From the second equation,

$$x^2=34-y^2,$$

and from the first, we have

$$x^2-y^2+\sqrt{(x^2-y^2)}=20,$$

substituting the above value of x^2, and reducing,

$$34-2y^2+\sqrt{(34-2y^2)}=20,$$

$$\sqrt{(34-2y^2)}=2y^2-14;$$

squaring, $34-2y^2=4y^4-56y^2+196,$

reducing, $4y^4-54y^2=-162,$

$$\therefore y=3;\text{ and } x=5.$$

SECOND SOLUTION—*By Nassau, Long Island.*

The first equation multiplied by $x+y$, gives $x^2-y^2+(x^2-y^2)^{\frac{1}{2}}=20$ a quadratic which determines

$$x^2-y^2=16.$$

This, added to, and taken from the second, gives

$$2x^2=50 \text{ and } 2y^2=18;$$

$$\therefore x=5,\text{ and } y=3.$$

THIRD SOLUTION.—*By Mr. John F. Jenkins, Middleton Academy, N. J.*

From the first equation, $x^2-y^2+\overline{x+y}\sqrt{\left(\frac{x-y}{x+y}\right)}=20$; the third term of which, when reduced to its most simple form, $=\sqrt{x^2-y^2}$. We have then, $\overline{x^2-y^2}+\sqrt{\overline{x^2-y^2}}=20$; a quadratic which gives $x^2-y^2=16$: and from first equation $x^2+y^2=34$ $\therefore$ $x^2=25$, $y^2=9$, and 5 and 3 are the values.

Messrs. James and Gerardus B. Docharty, L. I.· Thomas Mooney, junr. Brooklyn, L. I.; P. Barton, junr. Schenectady; and N. Vernon, Fredericktown, Md.; also sent ingenious solutions to this question.

QUESTION II. (223.)—*By Analyticus, New-York.*

A father dying, bequeaths an estate of ten thousand dollars in the following manner: in case that his wife (who was pregnant at the time of his death) should have a daughter, the wife was to have $\frac{2}{3}$ and the daughter $\frac{1}{3}$; but in case of a son, the wife was to have $\frac{1}{3}$ and the son $\frac{2}{3}$. Now, it so happened that the wife brought forth two sons and one daughter. Required, the respective shares of the wife, sons, and daughter.

FIRST SOLUTION.—*By Analyticus, New-Jersey*

This question may be considered in two ways. First, by supposing the son's, mother's, and daughter's shares to be as the numbers 4, 4, 2, 1, whence \$3636$\frac{4}{11}$= the share of each son, \$1818$\frac{2}{11}$= the mother's share, and \$909$\frac{1}{11}$= the daughter's do.; this solution supposes that a son should have twice as much as the mother, and the mother twice as much as the daughter.

Second, by supposing the shares to be as 2, 2, 2, 1; \$2857$\frac{1}{7}$= the share of each son and the mother separately, and \$1428$\frac{4}{7}$= the daughter's share; this solution is on the principle that no provision was made by the father for more than one son, and therefore they ought to have equal shares, whose sum should be twice as great as the mother's, &c.

SECOND SOLUTION.—*By Mr. James Docharty, L. I.*

Let x represent the daughter's share,

then $2x$=mother's,

and $4x$=each of the sons',

we have $2.4x+2x+x=11x=10000$

$$x=909.09.\tfrac{1}{11}$$
$$2x=1818.18.\tfrac{2}{11}$$
$$4x=3636.36.\tfrac{4}{11}.$$

Third Solution.—*By Mr. Thomas Mooney, junr. Brooklyn, L. I.*

It is evident, that the mother is to have twice as much as the daughter, and each son twice as much as the mother.

If $x=$ the daughter's share, $2x=$ the mother's share, and $4x=$ each son's share, therefore, $x=2x+4x+4x=10000$, *i. e.* $11x=10000$; consequently, $x=909\frac{1}{11}=$ the daughter's share, $2x=1818\frac{2}{11}=$ the mother's share, $4x=3636\frac{4}{11}=$ each son's share.

Many equally ingenious solutions, and various hypotheses were received from our correspondents, Messrs. N. Vernon, P. Barton, junr. P. E. Miles, O. Root, and Omicron, N. C. Mr. Vernon suggested the propriety of the WIDOW'S THIRD.

Question III. (224.)—*By Mr. Wm. Lenhart, York, Penn.*

The fractions $\frac{2}{3}$ and $\frac{1}{3}$ are such that their difference is equal to the sum of their cubes: are there other fractions having the same properties?

First Solution.—*By Mr. Benjamin Peirce, Mathematical Instructor at Harvard University, Cambridge, Mass.*

Let the fractions be $s+d$, and $s-d$.

We have $(s+d)^3+(s-d)^3=(s+d)-(s-d)$

or $s^3+3d^2s=d.$

Let $s=ad$

$$(a^3+3a)d^2=1$$

$$a^3+3a=\square \text{ and } a>1.$$

But if we should find $a'<1$, and such that $a'^3+3a'=\square$, we might make $a=\dfrac{3}{a'}$;

for $$\left(\frac{3}{a'}\right)^3+3\left(\frac{3}{a'}\right)=\frac{9}{a'^4}(3a'+a'^3)=\square.$$

Let $$b^3+3b=c^2$$

$$a=x+b, \text{ and } a^3+3a=(a'x+c)^2.$$

Hence, $x^3+(3b-a'^2)x^2+(3b^2-2a'c+3)x+b^3+3b-c^2=0.$

Let $3b^2-2a'c+3=0.$ Or, $a'=\dfrac{3b^2+3}{2c}$

we obtain $$x=a'^2-3b$$

$$a=a'^2-2b=\frac{(b^2-3)^2}{4b(b^2+3)}$$

By this formula, we can find a new value of a, from a given one which werepresent by b. Now, we may assume 1 for this value, since

$$1+3=4=\square.$$

But $b=1$ gives $a=\frac{1}{4}$, which is less than 1.

Instead of this value of a, we must use $a'=\frac{3}{a}=12$

Then $$d=\frac{1}{\sqrt{a'^3+3a'}}=\tfrac{1}{42}$$
$$s=a'd=\tfrac{12}{42},$$

So that $s+d=\frac{13}{42}$, and $s-d=\frac{11}{42}$ are fractions that satisfy the requisite conditions.

By making $b=\frac{1}{4}$, we might easily find another value of a, which we might again substitute for b. Proceeding in this way, we should find an infinite number of fractions possessed of the desired property.

SECOND SOLUTION.—*By Mr. C. Gill, Teacher of Mathematics, Sawpits, New-York.*

Let x and y be the two fractions having the required property, or $x-y=x^3+y^3=(x+y)(x^2-xy+y^2)=\frac{1}{4}(x+y)\cdot\left\{3(x-y)^2+(x+y)^2.\right\}$

Then, $(x-y)^2-\frac{4}{3}\cdot\frac{x-y}{x+y}=-\frac{1}{3}(x+y)^2$, and completing the square,

we have $\left\{x-y-\frac{2}{3(x+y)}\right\}^2=\frac{4-3(x+y)^4}{9(x+y)^2}=\square \therefore 4-3(x+y)^4$ $=\square$ (1).

$x+y=\pm 1$, is a partial solution of this formula: and comprises the example given in the question. For, if $x+y=1$, $x-y=\frac{1}{3}\therefore x=\frac{2}{3}$, and $y=\frac{1}{3}$; from this, others may be obtained in the following manner.

Let $x+y=\frac{m-n}{m+n}$, and substituting this in the formula (1), we have, after multiplying it, by the square $(m+n)$,[4]
$$m^4+28m^3n+6m^2n^2+28mn^3+n^4=\square; \qquad (2)$$
make it $=(-m^2+14mn+n^2)^2=m^4-28m^3n+694m^2n^2+28mn^3+n^4$.

Then $\frac{m}{n}=\frac{47}{14}$, $\therefore$ $x+y=\frac{33}{61}$, and $x-y=\frac{27}{671}$.

Hence, $x=\frac{195}{671}$, and $y=\frac{168}{671}$, which are two other fractions having the required conditions.

Make (2) $=(m^2+14mn-95n^2)^2=m^4+28m^3n+6m^2n^2-28.96mn^3+95^2n^4$.

Then, $\frac{m}{n}=\frac{4513}{1344}$; $x+y=\frac{3169}{5857}$, and $x-y=\frac{763585}{18560833}$; .

$\therefore$ $x=\frac{5403073}{18560833}$, and $y=\frac{4639488}{18560833}$, which are also such fractions as required.

More may be made by making $x+y=\frac{m-\frac{33}{61}n}{m+\frac{33}{61}n}=\frac{61m-33n}{61m+33n}$,

$x+y=\frac{5857m-3169n}{5857m+3169n}$, &c. ad infinitum.

ANOTHER SOLUTION.—*By the same.*

Let A B, be two fractions of the required kind, and $mv+A$, $nv+B$ two others possessing similar properties, then

$(m-n)v+A-B=(m^3+n^3)v^3+3(m^2A+n^2B)v^2+3(mA^2+nB^2)v+A^3+B^3$; put $A-B=A^3+B^3$, by the hypothesis $\therefore$

$$v^2+3\cdot\frac{m^2A+n^2B}{m^3+n^3}\cdot v=\frac{m-n-3(mA^2+nB^2)}{m^3+n^3},\ \therefore\left\{v+\tfrac{3}{2}\cdot\frac{m^2A+n^2B}{m^3+n^3}\right\}^2$$

$$=\frac{9}{4}\left(\frac{m^2A+n^2B}{m^3+n^3}\right)^2+\frac{m-n-3(mA^2+nB^2)}{m^3+n^3}=\square$$

Multiply this by the square $4(m^3+n^3)^2$; and in the product substitute $z-n$ for m, there results,

$$9(A+B)^2n^4-12(2+3B^2+3AB)n^3z+18(2-A^2+AB+2B^2)n^2z^2-4(5-3A^2+3B^2)n^3z^3+(4-3A^2)z^4=\square.$$

$$\text{Suppose it }=\left\{3(A+B)n^2-2.\frac{2+3B^2+3AB}{A+B}nz+\frac{9(2-A^2+AB+2B^2)(A+B)^2-2(2+3B^2+3AB)}{3(A+B)^3}.z^2\right\}^2$$

which equality gives $\frac{z}{n}=$to a formula.*

From these formulas, numbers *ad infinitum* may be had by substituting the last found numbers, for A and B.

If $A=\frac{2}{3}$, and $B=\frac{1}{3}$, $\frac{z}{n}=6$; $\frac{m}{n}=5$, $v=\frac{-1}{14n}$, and the numbers are $\frac{13}{42}$ and $\frac{11}{42}$.

Making $A=\frac{13}{42}$ and $B=\frac{11}{42}$, we get other numbers, &c. &c.

NOTE 1. Taking again the equation,

$$(m^3+n^3)v^2+3(m^2A+n^2B)v=(m-n)-3(mA^2+nB^2).$$

And making $m=-n$, $v=\frac{3(A^2-B^2)-2}{3n(A+B)}$; and the numbers are

$B+\frac{2}{3(A+B)}$ and $A-\frac{2}{3(A+B)}$.

When these formulas give one of the numbers negative, they are such (both taken positively) that their sum = the difference of their cubes. Thus, if $A=\frac{13}{42}$, and $B=\frac{11}{42}$, the numbers $\frac{10}{7}$ and $\frac{6}{7}$ are such, &c. &c.

NOTE 2. If also in the same equation. $m-n=3(mA^2+nB^2)$ or $\frac{m}{n}=\frac{1+3B^2}{1-3B^2}$, $v=-\frac{1-3A^2}{n}\cdot\frac{A(1+3B^2)^2+B(1-3A^2)^2}{(1+3B^2)^3+(1-3A^2)^3}$, and the numbers are $\frac{-2(A-B)(1-3A^2)^3}{(1+3B^2)^3+(1-3A^2)^3}$, and $\frac{2(A-B)(1+3B^2)}{(1+3B^2)^3+(1-3A^2)^4}$, which will in general possess similar properties.

* Mr. Gill, in his solution, furnished the formula, but it contained a fraction, the numerator and denominator of which consisted of so many terms, that it could not be printed in one line of the present page. It was, therefore, omitted. ED.

THIRD SOLUTION.*—*By Mr. Gerardus B. Docharty, L. I.*

Let $\frac{a}{d}$ and $\frac{c}{d}$ denote two fractions. Then, by the requisitions of the question, $\frac{a-c}{d}=\frac{a^3+c^3}{d^3}$ $\therefore$ by clearing of fractions, we have $d^2=\frac{a^3+c^3}{a-c}$. It now remains to render the latter expression a square, which evidently takes place when $a=2c$. Put $\therefore$ $a=2c-m$, then by substituting $\frac{9c^3-12mc^2+6m^2c-m^3}{c-m}$ must be a square. Assume its side $=3c+m$. Involving and reducing, $9mc^2=11m^2c$ $\therefore$ $c=\frac{11m}{9}$. Now, m may be any number whatever. If $m=1$, $c=\frac{11}{9}$, $a=\frac{13}{9}$, $d=\frac{42}{9}$. And the fractions will be $\frac{13}{42}$ and $\frac{11}{42}$; or $\frac{195}{671}$, and $\frac{168}{671}$, which will be two other fractions having the same properties; so on *ad infinitum*.

The required fractions as given in the above solution, agree with the results obtained by *all* our correspondents. This question is useful, as are all simple Diophantine problems. EDITOR.

QUESTION IV. (225.)—*By the same.*

Given $\begin{cases}(x+y)(xy+1)=18xy\\(x^2+y^2)(x^2y^2+1)=208x^2y^2\end{cases}$

to determine x and y.

FIRST SOLUTION.—*By Mr. Wm. Lenhart, the proposer.*

Dividing the first equation by xy, and the second by x^2y^2, we find $(x+y)\ (1+\frac{1}{xy})=18$, and $(x^2+y^2).(1+\frac{1}{x^2y^2})=208$, or, making the actual multiplication, $x+y+\frac{1}{y}+\frac{1}{x}=18$ and $x^2+y^2+\frac{1}{y^2}+\frac{1}{x^2}=208$. Now put $x+\frac{1}{x}=v$ and $y+\frac{1}{y}=w$, then will $v+w=18$ & $v^2+w^2=212$. Hence, by the usual methods, $v=4$ and $w=14$; i. e. $x+\frac{1}{x}=4$ and $y+\frac{1}{y}=14$; and by quadratics $x=2\pm\sqrt{3}$ and $y=7\pm4\sqrt{3}$.

* The solution of Mr. P. Barton, junr. was exactly similar to this.

SECOND SOLUTION—*By Mr. Samuel Ward, 3d, New-York*

Let the first equation be divided by xy, the second by x^2y^2 and the resulting ones being developed, we shall have

$$x+\frac{1}{x}+y+\frac{1}{y}=18$$

$$x^2+\frac{1}{x^2}+y^2+\frac{1}{y^2}=208$$

these being particular forms of reciprocal equations, we shall proceed after the manner of *Lacroix* in his *Complement D'Algébre*, and write in them x' for $x+\frac{1}{x}$ and y' for $y+\frac{1}{y}$, we shall then have

$$x'+y'=18$$
$$x'^2+y'^2=212$$

the latter subtracted from the square of the former, and divided by 2, gives

$$x'y'=56$$

having thus obtained the sum and product of x' and y', we readily find

$$x'=14 \text{ or } 4 \text{ and } y'=4 \text{ or } 14$$

$$\text{wherefore, } x=7\pm4\sqrt{3} \text{ or } 2\pm\sqrt{3}$$
$$y=2\pm\sqrt{3} \text{ or } 7\pm4\sqrt{3}.$$

THIRD SOLUTION—*By Mr. N. Vernon, Fredericktown, Md.*

Proceeding as in the first solution. Let $x+\frac{1}{x}=m$, $y+\frac{1}{y}=n$, then will $m^2=x^2+2+\frac{1}{x^2}$; $n^2=y^2+2+\frac{1}{y^2}$; & $m^2+n^2=212$. Again, $(m+n)^2=(18)^2=m^2+2mn+n^2=324$, from this subtract $m^2+n^2=212$, we get $2mn=112$, and this subtracted from $m^2+n^2=212$, leaves $m^2-2mn+n^2=100$, from which we find $m-n=10$; consequently, $m=14$ & $n=4$. Therefore, $x+\frac{1}{x}=14$ & $y+\frac{1}{y}=4$. From which we find

$x=7\pm4\sqrt{3}$ and $y=2\pm\sqrt{3}$.

Messrs. C. Gill, P. Barton, jun. E. Loomis, and several other ingenious correspondents, solvéd this question by a Biquadratic.—[ED.

QUESTION V. (226.)—*By the same.*

Given $axy=x^2-by^2$, and $x^3y^3=x^2+by^2$, to determine x and y, in terms of a and b by simple equations; and find general expressions for a and b which shall render x and y rational.

FIRST SOLUTION.—*By the proposer.*

By squaring both equations, and taking their difference, there re-

sults $x^6y^6-a^2x^2y^2=4bx^2y^2$, or $x^4y^4=a^2+4b$, hence $xy=(a^2+4b)^{\frac{1}{4}}$. This, substituted in the original equations, gives $x^2-by^2=a(a^2+4b)^{\frac{1}{4}}$ and $x^2+4by^2=(a^2+4b)^{\frac{3}{4}}$, or by addition $2x^2=a(a^2+4b)^{\frac{1}{4}}+(a^2+4b)^{\frac{3}{4}}$, from which $x=\frac{[a(a^2+4b)^{\frac{1}{4}}+(a^2+4b)^{\frac{3}{4}}]^{\frac{1}{2}}}{2}$ and thence y. But x and y are rational, therefore $xy=(a^2+4b^2)^{\frac{1}{4}}$ must also be rational. This may be effected by comparing a^2+4b with the formula $(r^2+n)^4=r^8+4r^6n+6r^4n^2+4r^2n^3+n^4=(r^8-2r^4n^2+n^4)+4r^2n(r^4+2r^2n+n^2)$; for if $a=r^4-n^2$ and $b=r^2n(r^2+n)^2$, we shall have $xy=r^2+n$, and by substitution in the above surd expression for x, we find $x=r(r^2+n)$ and consequently $y=\frac{1}{r}$. If $r=2$ and $n=1$ then will $a=15$ and $b=100$; and these are the least whole numbers that will make x and y rational: viz. $x=10$ and $y=\frac{1}{2}$.

Second Solution.—*By Mr. Benjamin Peirce, Cambridge.*

1. The sum of the given equations is $2x^2=x^3y^3+axy$
their difference is $2by^2=x^3y^3-axy$
the product of these is $4bx^2y^2=x^2y^2(x^4y^4-a^2)$

$$4b=x^4y^4-a^2$$
$$xy=(4b+a^2)^{\frac{1}{4}}$$
$$2x^2=((4b+a^2)^{\frac{1}{2}}+a)(4b+a^2)^{\frac{1}{4}}$$
$$x=(\tfrac{1}{2}(4b+a^2)^{\frac{1}{2}}+\tfrac{1}{2}a)^{\frac{1}{2}}(4b+a^2)^{\frac{1}{8}}$$
$$y=(\frac{1}{2b}(4b+a^2)^{\frac{1}{2}}-\frac{a}{2b})^{\frac{1}{2}}(4b+a^2)^{\frac{1}{8}}$$

2. If x and y are rational, their product must be so too,

or $(4b+a^2)^{\frac{1}{4}}$ must be so,
or $4b+a^2=d^4$

$$b=\tfrac{1}{4}d^4-\tfrac{1}{4}a^2$$
$$2x^2=d^3+ad$$
$$4x^2=2d^3+2ad=\square=e^2$$
$$a=\frac{e^2}{2d}-d^2$$

These values of a and b give $xy=d$, $x=\frac{e}{2}$ and $y=\frac{2d}{e}$.

Third Solution.—*By Mr. C. Gill, Sawpits, New-York.*

Take the difference of the squares of the given equations,

then $$x^6y^6-a^2x^2y^2=4bx^2y^2$$

$$\therefore\ xy=(a^2+4b)^{\frac{1}{4}}$$

By substitution, $$\begin{cases} x^2+by^2=(a^2+4b)^{\frac{3}{4}} \\ x^2-by^2=a(a^2+4b)^{\frac{1}{4}} \end{cases}$$

Hence $$x=\sqrt{\tfrac{1}{2}(a^2+4b)^{\frac{3}{4}}+\tfrac{1}{2}a(a^2+4b)^{\frac{1}{4}}}$$

$$y=\sqrt{\frac{1}{2b}(a^2+4b)^{\frac{3}{4}}-\frac{a}{2b}(a^2+4b)^{\frac{1}{4}}}$$

To make x & y rational, let $b=av+v^2$; then $(a^2+4b)^{\frac{1}{4}}=(a+2v)^{\frac{1}{2}}$.

Make $a=n^2-2v$, then $(a^2+4b)^{\frac{1}{4}}=n$, and $b=n^2v-v^2$ $\therefore$

$$x=\sqrt{n(n^2-v)};\quad y=\sqrt{\frac{n}{(n^2-v)}}$$

Let $n(n^2-v)=n^2s^2$, then $v=n^2-ns^2$, hence

$a=2ns^2-n^2$, $b=n^3s^2-n^2s^4$, $x=ns$, and $y=\frac{1}{s}$

$n=288^{\frac{1}{4}}$, $s=\sqrt{\frac{6+288^{\frac{1}{2}}}{2\cdot 288^{\frac{1}{4}}}}$, $x=6$, $b=63$,

× ∴ n & s are not rational ∴ x & y are [illegible]

Question VI. (227.)—*By Mr. O. Root, Vernon, N. Y.*

Given, the base and sum of the sides of a plane triangle to construct it when the angles at the base are as 1 to 2.

First Solution.—*By the Proposer.*

With the middle of the base as centre, and with a transverse axis equal the sum of the sides, describe an ellipse; also, with the origin at the extremity of the given base, describe an hyperbola, whose semitransverse equals $\frac{1}{3}$ of the base, and semiconjugate equals the base. The intersection of these curves will determine the vertex of the triangle. For when the base and sum of the sides are given, the vertex will be in the ellipse, and when the base angles are as 1 : 2, the vertex will be on the hyperbola $\therefore$ by both conditions the vertex will be found at their intersection.

Again, put $b=$ the given base, and $s=$ the sum of the sides, $x=$ the side opposite the largest angle at the base, and $(s-x)=$ the other side, let $y=$ the line that bisects the largest base angle and terminates on the side (x). From similar triangles, we have $x:b::(s-x):y$,

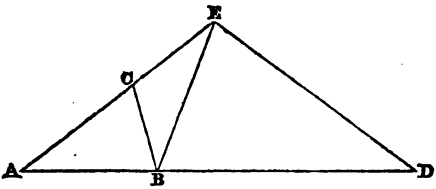
× ∴ $n=4$, $s=\frac{3}{2}$, $a=2$, $b=63$, $x=6$.

whence eliminating y and solving, and $(x-y) : y :: (s-x) : b$, for x we find $x=\frac{s(s+b)}{(2s+b)}$ and $(s-x)=\frac{s^2}{2s+b}$; which are the sides required.

The solution of Mr. E. Loomis, by a property of the ellipse, was also extremely ingenious. Ed.

Second Solution.—*By Theodore Strong, Professor of Mathematics, &c. Rutger's College, New-Brunswick, N. J.*

Let AB= the given base, produce AB to D, so that BD may equal



the sum of the other sides of the triangle; then from A and D as centres, with radii =BD describe arcs intersecting at E, join AE, DE, BE, and draw BC making the angle EBC=BEC, then ABC is the triangle required. For (since DE=DB,) the angle DEB=DBE and BEC=EBC ∴ AED=DBC; hence, and by Euc. B. 1. prop. 13. 32. ABC=EAD+EDA=2CAB (since EA=ED); but (since CBE=CEB,) CB=CE ∴ AC+CB=AE=BD.

Messrs. C. Gill and N. Vernon also gave ingenious geometrical solutions of this question. But by far the greater portion of our correspondents obtained the expression (calling x the smaller angle, b the base, and s the sum of the sides) $\cos. x=\frac{b+s}{2s}$, which, as it has been observed by Mr. Benjamin Peirce, may be thus constructed: On $2s$ as a diameter, describe a circle. At the extremity of this diameter, draw two chords each equal to $b+s$, and from the same point set off on one of them the base b. At the middle of b erect a perpendicular to meet the diameter. Join the extremity of b with this point of intersection by a line, which produced, will form, with the two chords, the required triangle. Ed.

Question VII. (228.)—*By Mr. Benjamin Wiggins.*

In a given parabola, it is required to determine the greatest inscribed ellipse, one of whose axes shall be parallel to the base of the parabola.

First Solution.—*By Mr. P. E. Miles, Buffalo, N. Y.*

Let ABC be the given parabola, whose semi-base, AD$=b$, and altitude, DC$=a$. Assume a point F, in the arc AC, of the semi-parabola ADC, as the point of its contact with the inscribed ellipse, and draw the ordinate FG, perpendicular to, and meeting DC in G. Draw, also, the tangent FE, meeting DC produced in E. Put $x=$EC, which also by (Hutton's Con. Sec. Th. 5. of the Parab.) $=$CG. Then, by Th. 9. will $b\sqrt{\frac{x}{a}}=$FG. Also, by Th. 7. of the Ellipse, $\frac{a^2-x^2}{a}=$ DK $=$ that diameter of the ellipse, perpendicular to the base of the parabola. From which it is plain, that CK$=a-\frac{a^2-x^2}{a}=\frac{x^2}{a}$, GK $=x-\frac{x^2}{a}=\frac{x(a-x)}{a}$, and GH$=\frac{(a-x)^2}{2a}$, H being the centre of the ellipse. Now, by Th. 2. of the Ellipse, DG$\times$GK : (KH)^2 : : (FG)^2 : (IH)^2, I, being at the extremity of that axis of the ellipse, which is parallel to the base of the parabola. That is,

$$(a-x)\times\frac{x(a-x)}{a}:\left(\frac{a^2-x^2}{2a}\right)^2::\frac{b^2x}{a}:\frac{b^2}{4a^2}(a+x)^2=(\mathrm{IH})^2,\text{ and }\frac{b}{2a}(a+x)=\mathrm{IH}.$$

Therefore, the diameters are $\frac{a^2-x^2}{a}$, and $\frac{b}{a}(a+x)$; whose product multiplied by $c=(.7854)$, gives $\frac{bc}{a^2}(a^3+a^2x-ax^2-x^3)$ $=$ the area $=$ the maximum by the question.

And, by putting its differential $=$ zero, and dividing by $\frac{bc}{a^2}dx$, we have, $a^2-2ax-3x^2=0$; or $3x^2+2ax=a^2$, and $x=\frac{a}{3}$. Wherefore, the two required diameters are $\frac{8a}{9}$, and $\frac{4b}{3}$; the former being perpendicular, and the latter parallel to the base of the parabola. It may here be observed, that when b is to a, in the ratio of 2 : 3, the maximum inscribed ellipse becomes a circle; but, if the relative value of b be *greater* or *less* than this, the figure becomes an ellipse, whose transverse axis is, accordingly, *parallel* or *perpendicular* to the base of the parabola.

Second Solution.—*By Analyticus, New-Jersey.*

Let $px'=y^2$ (1) be the equation of the parabola, having h for its

height and b for its base; also let $y^2+mx^2=k^2$(2) be the equation of the ellipse, whose semi-axes are $k, \frac{k}{\sqrt{m}}$: then $\frac{k^2}{\sqrt{m}}=$max. Now, because y is to be the same in (1) and (2), and that the ellipse is to touch the middle of b, at the extremity of the semi-axis $\frac{k}{\sqrt{m}}$, I have $x'=h-\left(\frac{k}{\sqrt{m}}+x\right)$, also since the ellipse touches the parabola at the extremity of y, $x=\frac{p}{2m}$; hence $x'=h-\left(\frac{k}{\sqrt{m}}+\frac{p}{2m}\right)$; by substituting the values of x', x in (1) and (2), then putting the values of y^2, equal to each other, there results the equation $p\left(h-\frac{k}{\sqrt{m}}-\frac{p}{2m}\right)$ $=k^2-\frac{p^2}{4m}$ or $\sqrt{m}=\frac{p}{2(\sqrt{hp}-k)}$, hence $\frac{k^2}{\sqrt{m}}=\frac{2k^2}{p}\left(\sqrt{hp}-k\right)=$ max. $\therefore k=\frac{2\sqrt{hp}}{3}=\frac{b}{3}$, $\sqrt{m}=\frac{3}{2}\sqrt{\frac{p}{h}}$, and $\frac{k}{\sqrt{m}}=\frac{9}{4}h$.

THIRD SOLUTION.—*By L'Inconnu, Cincinnati.*

Let the origin of the co-ordinates be at the centre of the base of the given parabola, then its equation will be $y^2=p(h-x)$, h being the height; that of the ellipse summit at the origin is

$$y^2=\frac{b^2}{a^2}(2ax-x^2);$$

since these two curves have a common point, we have

$$p(h-x)=\frac{b^2}{a^2}(2ax-x^2)\ (1).$$

The expression for the subtangent of the parabola is $2(h-x)$ and that of the ellipse is

$$\frac{2ax-x^2}{x-a}.$$

In order that these curves may touch, we shall have

$$2(h-x)=\frac{2ax-x^2}{x-a};$$

from this we find

$$a=\frac{2hx-x^2}{2h}\ (2);$$

this substituted in (1) gives

$$b=\frac{2hx-x^2}{2b}\cdot\sqrt{\frac{ph}{x^2}}\ (3);$$

Now the area of the ellipse is

$$4\pi ab = 4\pi\left(\frac{2hx-x^2}{2h}\right)\frac{^2\sqrt{ph}}{x^2};$$

and its square neglecting constants and reducing, becomes

$$\left(\frac{2hx-x^2}{x^2}\right)^4 = \text{a maximum};$$

the differential of this, being made $=0$, gives $x=\frac{2}{3}h$; this substituted in equation (2) gives $a=\frac{4}{9}h$ and $2a=\frac{8}{9}h$: and in(3)gives $b=\frac{2}{3}h\sqrt{p}$ and $2b=\frac{4}{3}h\sqrt{p}$ the maximum ellipse is therefore determined.

It has been properly observed by Omicron, N. C., that "this is question 223. vol. 2. p. 25. of Davis's edit. of the Gent. Diary, where, taking $a=$ altitude of the least triangle circumscribing the parabola, ab its base, and x the minor axis, it is shown that $x=\frac{2h}{3}$, and the axis major $=\frac{2b}{\sqrt{3}}$."

Ed.

Question VIII. (229.)—*By Mr. James Macully, Richmond, Va.*

In a given ellipse, it is required to inscribe the greatest possible equilateral triangle.

First Solution.—*By Mr. Benjamin Peirce, junr. Cambridge, Mass.*

Let the ellipse be $A^2y^2+B^2x^2=A^2B^2$.

Let the centre of the triangle be x',y'.

The equation of the ellipse referred to this point as origin, and to polar co-ordinates is,

$$A^2(r \sin. w+y')^2+B^2(r \cos. w+x)^2=A^2B^2$$

where the axis of w is parallel to A. This, developed, becomes $(A^2 \sin.{}^2w+B^2 \cos.{}^2w)r^2+(2A^2y' \sin. w+2B^2x' \cos. w)r+A^2y'^2+B^2x'^2-A^2B^2=0$.

Let the vertices of the triangle be w', $w''=w'+120^\circ$, and $w'''=w'-120^\circ$.

Let r' be the radius vector for all.

Subtract the equation in w''' from that in w''

$(A^2-B^2)(\sin.{}^2w''-\sin.{}^2w''')r'+2A^2y'(\sin. w''-\sin. w''')+2B^2x'(\cos. w''-\cos. w''')=0$.

But $\quad \sin. w''+\sin. w'''=2 \sin. w' \cos. 120^\circ=-\sin. w'$

$\sin. w''-\sin. w'''=2 \cos. w' \sin. 120^\circ=\sqrt{3} \cos. w'$

$\cos. w''+\cos. w'''=+2 \cos. w' \cos. 120^\circ=-\cos. w'$

$\cos. w''-\cos. w'''=-2 \sin. w' \sin. 120^\circ=-\sqrt{3} \sin. w'$.

Hence, $-(A^2-B^2)$ sin. w' cos. $w'r'+2A^2$ cos. $w'y'-2B^2$ sin. $w'x'=0$.
Instead of w', we might write in this equation w'' and w'''.

From that in w'' multiplied by cos. w''', take that in w''' multiplied by cos. w''. Also from that in w'' multiplied by sin. w''', take that in w''' multiplied by sin. w''. We thus obtain,

$$-(A^2-B^2)(\sin. w''-\sin. w''')\cos. w''\cos. w'''r'-2B^2x'(\sin. w''\cos. w'''-\sin. w'''\cos. w''-(A^2-B^2)(\cos. w''-\cos. w''')\sin. w''\sin. w'''r'+2A^2y'(\sin. w'''\cos. w''-\sin. w''\cos. w''')=0.$$

Or by the above equations,

$$-\sqrt{3}(A^2-B^2)\cos. w'\cos. w''\cos w'''r'-2B^2x'\sin. (w''-w''')=0$$

$$\sqrt{3}(A^2-B^2)\sin. w'\sin. w''\sin. w'''r'-2A^2y'\sin. (w''-w''')=0.$$

But $\sin. (w''-w''')=\sin. 240^\circ=-\frac{1}{2}\sqrt{3}$,
our equations are then, finally,

$$B^2x'=(A^2-B^2)\cos. w'\cos. w''\cos. w'''.r'$$

$$A^2y'=-(A^2-B^2)\sin. w'\sin. w''\sin. w'''.r'$$

These values of x' and y', substituted in the polar equation in w', give, after reduction, an equation of the following form,

$$(P\sin.^6 w'+Q\sin.^4 w'+R\sin.^2 w'+M)r'^2+N=0.$$

In the differential of this equation, r' being a maximum, dr' must be equal to zero. This gives for one of the vertices of each of the two, the maximum and the minimum triangles.

$$\sin. w'\cos. w'=0.$$

So that either $w'=0$ and $y'=0$,
or $w'=90^\circ$ and $x'=0$.

Both of these triangles then have a vertex at the extremity of one of the axes, and their centre of gravity in this axis. To determine which is the maximum, we will have recourse to the equation referred to the extremity of the axis A.

$$A^2y^2+B^2x^2=B^2Ax.$$

The co-ordinates of the other vertices are,

$$y=\tfrac{1}{2}s \text{ and } x=\frac{\sqrt{3}}{2}s,$$

s being the side of the triangle.

Hence, $$s=\frac{2\sqrt{3}B^2A}{A^2+3B^2}$$

For the other triangle

$$s'=\frac{2\sqrt{3}A^2B}{B^2+3A^2}$$

The difference of these is $s'-s=\dfrac{2\sqrt{3}AB(A-B)^3}{(A^2+3B^2)(B^2+3A^2)}$.

If $A>B$, we must have $s'>s$, and of course the maximum sought.

Second Solution.—*By Mr. C. Gill, Sawpits, N. Y.*

Let x', y', ; x'', y'' ; and x''', y''' be the co-ordinates of any three points of the ellipse. The general equation gives

$$\left.\begin{array}{l} x'^2+n'^2y'^2=k^2 \\ x''^2+n^2y''^2=k^2 \\ x'''^2+n^2y'''^2=k^2 \end{array}\right\}\ldots.(A)$$

The squares of the lines connecting the three points, will be $(x'-x'')^2+(y'-y'')^2$, $(x'-x''')^2+(y'-y''')^2$, and $(x''-x''')^2+(y''-y''')^2$; and $\therefore$ if they are the angular points of an equilateral triangle,

$$\left.\begin{array}{l} (x'-x'')^2+(y'-y'')^2=(x'-x''')^2+(y'-y''')^2 \\ (x'-x'')^2+(y'-y'')^2=(x''-x''')^2+(y''-y''')^2 \end{array}\right\}\ldots(B)$$

Moreover, the triangle is a max. $\therefore$ the square of its side is a max. Equating the differentials of the above three equal values of it to zero, and substituting in them the values of dx', dx'', dx''', found by the differentiation of the three equations (A) we get

$$\left.\begin{array}{l} \dfrac{dy'}{dy''}=\dfrac{x'}{x''}\cdot\dfrac{n^2y''(x'-x'')-x''(y'-y'')}{n^2y'(x'-x'')-x'(y'-y'')} \\ \dfrac{dy'''}{dy'}=\dfrac{x'''}{x'}\cdot\dfrac{n^2y'(x'-x''')-x'(y'-y''')}{n^2y'''(x'-x''')-x'''(y'-y''')} \\ \dfrac{dy''}{dy'''}=\dfrac{x''}{x'''}\cdot\dfrac{n^2y'''(x''-x''')-x'''(y'-y''')}{n^2y''(x''-x''')-x''(y''-y''')} \end{array}\right\}\ldots.(C)$$

Multiply together the three equations in (C), there results an equation, which, with the equations (A) and (B), will determine the six unknown quantities.

It has been observed by Professor Strong, that if two straight lines are drawn from one of the extremities of the conjugate axis, on opposite sides of it, so as each to make an angle of 30° with the conjugate, that the triangle formed by joining the points where the straight lines meet, the ellipse again will be the triangle sought.

Let $h=$ the side of the triangle, $a=$ the semi-transverse axis, and $b=$ the semi-conjugate axis, then

$$h=\frac{4ba^2\sqrt{3}}{b^2+3a^2}.$$

Ed.

Third Solution.—*By L'Inconnu, Cincinnati.*

It is evident that only two equilateral triangles can be inscribed in an ellipse, one the maximum and the other the minimum ; each of these have a vertex at the extremity of one of the axes, as well as at its centre of gravity in this axis. The equation of the ellipse, origin at the extremity of the transverse axis, is $a^2y^2+b^2x^2=b^2ax(1)$. by an easy

process we shall obtain for the co-ordinates of the two other vertices of the inscribed triangle, (calling c either of its sides) the expression

$$c=\frac{2\sqrt{3}\,a^2b}{b^2+3a^2},\ c'=\frac{2\sqrt{3}\,b^2a}{a^2+3b^2}(3);$$

since a exceeds b, c exceeds c'; take the difference of equations (3) and we have the required maximum, which may be thus constructed. From the extremity of the conjugate axis, draw two lines, (in the ellipse) each making an angle of 30° with it; connect by a straight line the points in which these lines intersect the perimeter of the ellipse, and we have accurately the maximum required. By the same process, with reference to the transverse, we are enabled to construct the minimum.

Question IX. (230.)—*By Mr. Francis Sherry New-York.*

It is required to inscribe the greatest trapezoid in a given segment of a circle.

First Solution.—*By the proposer.*

By plane geometry, it is readily proved, that the trapezoid is a maximum, whose angular points are in the extremities of the chord and points of trisection of the arch: Hence the problem is resolved into the trisection of a circular arch, which is effected by the hyperbola, as follows. Trisect the given chord, and with one extremity as a focus, the adjacent point of trisection as a vertex, and the other point of trisection as the centre of the curve, describe an hyperbola, it will give one point of trisection of the given arch, or of any arch of a circle described on the same chord: hence the points of trisection of the arch being determined, the trapezoid is determined.

Second Solution.—*By Professor Strong, Rutgers' College.*

Let $2p'$=the arc of the given segment and $2p$=the arc cut off by the parallel side of the trapezoid rad. (1)&R=the radius of the given circle: then 2R sin. p', 2R sin. p, are the parallel sides of the trapezoid & R(cos. p—cos. p')=their perpendicular distance; hence R^2 (sin. p'+sin. p).(cos. p—cos. p')=the area=max. ∴ cos. $2p$=cos. $(p'-p)$ or $p=\frac{p'}{3}$.

Third Solution.—*By Mr. C. Gill,* Sawpits, N. Y.*

Let a be the diameter of the circle of which the segment is a part, and let ϕ, δ be the angles made with a tangent at the extremity of the

* We announce with regret, the sudden death of this gentleman, which occurred about six months since.

segment by two sides of the trapezoid; one of which is evidently the chord of the segment, and consequently δ is a given angle. The equation of the circle, $v=a$ sin. θ, gives the parallel sides of the trapezoid, a sin. δ, and a sin. $\delta-2a$ sin. ϕ cos. $(\delta-\phi)$; and their perpendicular distance a sin. ϕ sin. $(\delta-\phi)$. Hence its area is a^2 (sin. δ—sin. ϕ cos. $\overline{\delta-\phi}$) sin. ϕ sin. $(\delta-\phi)=\frac{1}{2}a^2$ sin. $2\,\phi$ sin. $^2(\delta-\phi)=a$ max. Equating its differential to zero, we have sin. $(\delta-\phi)$ sin. $(\delta-3\phi)=0$. The roots of this equation are $\phi=\delta$, $\phi=\delta-180^\circ$ $\phi=\delta-360^\circ$ $\phi=\frac{1}{3}\delta$, $\phi=\frac{1}{3}\delta-60^\circ$, and $\phi=\frac{1}{3}\delta-120^\circ$. Of these, the fourth and fifth are the only applicable roots. The first and second giving a *minimum*, and the third and sixth impossible. When $\delta\angle 180^\circ$, the fifth is also impossible; and when $\delta > 180^\circ$, the two values for the area resulting from the roots $\phi=\frac{1}{3}\delta$, and $\phi=\frac{1}{3}\delta-60^\circ$, are $\frac{1}{2}a^2$ sin. $^3\,\frac{2}{3}\delta$, and $-\frac{1}{16}a^2$(sin. $\frac{2}{3}\delta+$cos. $\frac{2}{3}\delta\sqrt{3})^3$ and comparing these together, sin. $\frac{2}{3}\delta\ x-\frac{1}{2}$(sin. $\frac{2}{3}\delta+$cos. $\frac{2}{3}\delta\sqrt{3}$) or sin. $\frac{2}{3}\delta\sqrt{3}\ x$ — cos. $\frac{2}{3}\delta$. In all cases from $\delta=0^\circ$ to $\delta=225^\circ$, sin. $\frac{2}{3}\delta\sqrt{3} >$ —cos. $\frac{2}{3}\delta$, but from $\delta=225^\circ$ to $\delta=360^\circ$ sin. $\frac{2}{3}\delta\sqrt{3}\angle$—cos. $\frac{2}{3}\delta$. Hence, when $\delta\angle 225^\circ$, $\phi=\frac{1}{3}\delta$; when $\delta=225^\circ$, $\phi=75^\circ$, or 15°, the two resulting trapezoids being *equal;* and when $\delta > 225^\circ$, $\phi=\frac{1}{3}\delta-60^\circ$.

I am not aware that any writer on the *Differential Calculus* has noticed this curious property of the variation of *maximum* or *minimum* values, corresponding to the variations in the indeterminates: the inquiry is curious and interesting.

Question X. (231.)—*By the same.*

Given

$$\left(\frac{\text{sin. }^2\phi}{\text{cos. }^2\phi}\right)^{\text{sec. }^2\phi}=\frac{4\text{ cos. }^2\phi}{3}(\text{tan. }^2\phi)^{\text{tan. }^2\phi}$$

to determine ϕ.

First Solution.—*By Omicron, N. C.*

$$\left(\frac{\text{sin. }^2\phi}{\text{cos. }^2\phi}\right)^{\text{sec. }^2\phi}=\tfrac{4}{3}\text{ cos. }^2\phi\,(\text{tan. }^2\phi)^{\text{tan. }^2\phi}=(\text{tan. }^2\phi)^{1+\text{tan. }^2\phi};$$

whence, tan. $^2\phi=\frac{4}{3}$ cos. $^2\phi=\frac{\text{sin. }^2\phi}{\text{cos. }^2\phi,}$ and 4 cos. $^4\phi=3$ sin. $^2\phi$, conse-

quently, $2\cos.^2\phi=\pm\sin.\phi\sqrt{3}$, therefore $\sin.\phi=\frac{1}{4}(\pm\sqrt{19}-\sqrt{3})=$ $41^\circ..2'..58''.8$.

Second Solution.—*By Mr. P. Barton, Schenectady, N. Y.*

Let y denote the natural tangent of the arc ϕ; then we shall have $(y^2)^{y^2+1}=\frac{4}{3+3y}\overset{+}{}(y^2)^{y^2}$ by the principles of trigonometry; this simplified, gives $\log. y^2+\log. \overline{3+3y^2}=\log. 4=,602060$; from which y is found by approximation $=,870809+$; to which corresponds $41^\circ\ 2'\ 58''=$the value of the arc ϕ.

Third Solution.—*By Mr. E. Loomis, Baltimore, Md.*

Substituting equivalent expressions $\left(\frac{\tan.^2\phi}{r^2}\right)^{r^2+\tan.^2\phi}=\frac{4}{3}$ $\frac{r^4}{r^2+\tan.^2\phi}\overline{\tan.^2\phi}^{\tan.^2\phi}$

Put $r=1$. $x=\tan.^2\phi$; $x^{x+1}=\frac{4x}{3x+3}^{x}$ or $x=\frac{4}{3x+3}$ which gives $x=.75830573=\tan.^2\phi$.

Hence, $\phi=41^\circ 2' 37'' 12'''$ nearly.

This is similar to the answer of the proposer, who found $\sin.\phi=$ .656712. nearly. Ed.

Question XI. (232.)—*By Cartesius, Cincinnati, Ohio.*

It is required, to divide a quadrant of a circle into three arcs which shall be to each other in the ratio of three given right lines.

First Solution.—*By Professor Strong, New-Brunswick, N. J.*

Let the radius of the quadrant be denoted by unity, and P= the length of its arc, and let the numbers m, n, p, be as the parts into which the arc is to be divided; then assume

$$(m+n+p)x=\text{P} \text{ or } x=\frac{\text{P}}{m+n+p},$$

and mx, nx, px, the parts required are found; or the parts can readily be found by finding x in the equation $\tan. mx\times\tan.(n+p)x=1$: or describe a common cycloid, with the circle to which the quadrant belongs as generating circle, then divide one fourth of the base of the cycloid into three parts, which are as m, n, p.

Second Solution.—*By Omicron, N. C.*

This question is, I presume, of equal antiquity with that of the duplication of the cube, and may be readily solved by means of the quadratrix in the following manner. Divide the vertical radius of the quadrant into parts which are to one another in the given ratio, and through the points of division draw perpendiculars meeting the quadratrix in three points; these radii drawn through these points will divide the quadrantal arc into the parts required.

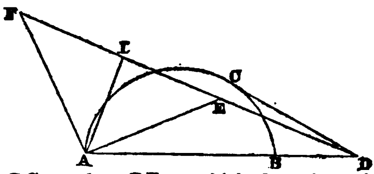
This question was, by a mistake, incorrectly printed, and should have been as follows: "Required to divide a quadrant into three arcs, whose *cosines* shall be to each other in the ratio of three given right lines." Ed.

Question XII. (233.)—*By Mr. William Lenhart.*

In a triangle, having given the base, the ratio of the sides and one of the angles adjacent to the base, double the vertical angle; to determine the triangle.

First Solution.—*By the Proposer.*

With a radius equal to the half of $\frac{n}{m}$, describe the circle ACB, and from any point C, in the circumference, draw a tangent CD, equal to unity. Through the centre of the circle draw DBA, make DE and EA each equal to DC, produce DE, to which, from A, apply AF equal to AE or CD, and FAD will be a triangle similar to the one required.



Demonstration.—Draw AI perpendicular to, and consequently bisecting, FE; and, since $AD\times DB = AD\times(AD-AB) = AD\times\left(AD - \frac{n}{m}\right) = \overline{AD}^2 - \frac{n}{m}AD = \overline{CD}^2 = 1$: we therefore have $\overline{AD}^2 = \frac{n}{m}AD + 1$. But $\overline{AD}^2 - \overline{AF}^2 = \overline{AD}^2 - 1 = \overline{DI}^2 - \overline{IF}^2 = FD\times DE = FD$, therefore $\overline{AD}^2 = FD + 1 = \frac{n}{m}AD + 1$ or $\frac{n}{m}AD = FD$. Hence, $FD : AD :: n : m$, and the angle AFD is evidently double that of ADF, consequently the triangle FAD is similar to the one required.

Ingenious geometrical solutions were also received to this question from Messrs. C. Gill; E. Loomis, Baltimore; and N. Vernon, Fredericktown, Md. Ed.

SECOND SOLUTION.—*By Mr. Benjamin Peirce, Cambridge, Mass.*

Let the base of the triangle be b, the ratio of the sides $\frac{c}{d}$, and the vertical angle ω.

$$\frac{\sin.\ 3\omega}{\sin.\ 2\omega}=\frac{c}{d}=\frac{3\cos.\ ^2\omega-\sin.\ ^2\omega}{2\cos.\ \omega}=\frac{4\cos.\ ^2\omega-1}{2\cos.\ \omega}$$

$$\cos.\ \omega=\frac{c}{4d}\pm\sqrt{1+\left(\frac{c}{d}\right)^2}$$

Let $$\frac{c}{2d}=\cot.\ a,$$

$$\cos.\ \omega=\tfrac{1}{2}\cot.\ a\pm\tfrac{1}{2}\operatorname{cosec}.\ a=\frac{\cos.\ a\pm1}{2\sin.\ a}=-\tfrac{1}{2}\tan.\ \tfrac{1}{2}a \text{ or } =\tfrac{1}{2}\cot.\ \tfrac{1}{2}a.$$

QUESTION XIII. (234.)—*By Mr. E. Giddens, Lockport, N. Y.*

The diameters of two circles are 2 and 5 respectively, and the distance between their centres is 10; to describe another circle touching the two and their tangent.

FIRST SOLUTION.—*By L'Inconnu, Cincinnati.*

Let r, r', be the radii of the two given circles $r \neq r'$; c the given distance of their centres, T their common tangent, whose expression will evidently be

$$T=\sqrt{c^2-(r-r')^2} \tag{1}$$

Let ρ denote the radius of the required circle, then we shall have

$$\sqrt{(r+\rho)^2-(r-\rho)^2}=\sqrt{4r\rho} \tag{2}$$

= that part of the common tangent, which is contained between the points of its contact with the circles whose radii are r and ρ. In like manner, $4r'\rho$=the other part of the tangent, we shall then have,

$$\sqrt{4\rho r}+\sqrt{4\rho r'}=T;$$

whence,

$$4r\rho+4r'\rho+8\rho\sqrt{rr'}=T^2, \text{ and } \rho=\frac{T^2}{4(r+r'+2\sqrt{rr'})}.$$

This, together with the value of T, substituted in equation (2) gives the position of its centre, $\rho=3.668$.

SECOND SOLUTION.—*By Analyticus, New-York.*

Let r' r'' denote the radii of the given circles $r'' \neq r'$.

Their common tangent is a right line given in position, which we shall assume for the axis of x, and a perpendicular thereto from its point of contact with the circle rad. r', will be the axis of y.

Let (x, y) be the co-ordinates of the centre of any circle touching the circle (rad. r') and the common tangent, and r its variable radius; we shall then have, by reason of the distance of the centres of these two circles,

$$(r+r')^2=x^2+(r-r')^2$$

whence, $$x^2=4rr' \qquad (1)$$

which becomes, if y be written for r, $x^2=4r'y$;

this is the equation of a common parabola, whose summit is at the point of contact of the first circle with the tangent, whose axis coincides with the axis of the ordinates, and whose parameter $=4r'$. This parabola being described, will be the *locus* of the centres of *all* the circles, touching the circle rad. r' and the common tangent.

Now, if we remove the origin to the point of contact of the circle rad. r'', with the same tangent, we shall have, in like manner,

$$x'^2=4rr'' \qquad (2)$$

which, in this case, becomes

$$x'^2=4r''y',$$

which is the *locus* of the centres of *all* the circles that will touch this circle and the tangent, and is, in fact, another parabola, whose summit is at the point of contact of the second circle, with the tangent and whose parameter $=4r''$; these two loci will intersect in at least one point, which will be the centre of the required circle.

Now, if equation (1) be divided by (2), and the square root of the result be taken, we have

$$\frac{x}{r'^{\frac{1}{2}}}=\frac{x'}{r''^{\frac{1}{2}}}\text{, whence } r''^{\frac{1}{2}} : r'^{\frac{1}{2}} :: x' : x. \qquad (3)$$

Let $d=x+x'=$ distance between the two given points of contact, then

$$r''^{\frac{1}{2}}+r'^{\frac{1}{2}} : r'^{\frac{1}{2}} :: d : x \therefore x=\frac{dr'^{\frac{1}{2}}}{r''^{\frac{1}{2}}+r'^{\frac{1}{2}}},$$

this, substituted in (1), gives

$$r=\frac{d^2}{4(r''^{\frac{1}{2}}+r'^{\frac{1}{2}})^2}=3{,}668\text{ nearly.} \qquad (4)$$

x and y (or r) being known, the tangent circle is determined in magnitude and position.

Note.—Equation (4) may also be constructed geometrically, by means of the right line and circle: for its denominator denoting a right line, d will be a mean proportioned between it and r.

Similar results were likewise obtained by Professor Strong, Messrs. Peirce, Gill, Omicron, E. Loomis, and several other correspondents.

Ed.

Third Solution—*By Mr. Samuel Ward, 3d, New-York.*

Let r', r'' and r be respectively the radii of the two given circles and the required one, and d that part of the tangent comprised between the points of contact of r' and r'', the origin of co-ordinates being at the point of contact of the first circle, its equation will be

$$x^2+y^2=2yr', \tag{1}$$

and that of the second will be

$$(x-d)^2+y^2=2r''y, \tag{2}$$

also

$$(r'+r)^2-(r'-r)^2=a^2, \tag{3}$$

a and r being the co-ordinates of the centre of the required circle.

$$(r''+r)^2-(r''-r)^2=(d-a)^2$$

(3) gives

$$4rr'=a^2, \tag{4}$$

whence

$$4rr''=(d-a)^2.$$

If the latter be divided by the square of the former, and the square root taken, we shall have

$$\frac{r'^{\frac{1}{2}}}{r''^{\frac{1}{2}}}=\frac{a}{d-a} \text{ or } t=\frac{a}{d-a};$$

whence,

$$a=\frac{d}{t+1}, \text{ in which } d=\sqrt{D^2-(r''-r')^2}$$

D being the distance of the centres of the two given circles, and by (4)

$$r=\frac{d^2}{4r'}=3{,}668 \text{ nearly},$$

whence the tangent circle is determined.

Question XIV. (235.)—*By Cartesius, Cincinnati, Ohio.*

Required, the locus of the intersections of the ordinates of an ellipse, with the perpendiculars let fall from its centre upon the tangents at the extremities of those ordinates.

First Solution.—*By Mr. Samuel Ward, 3d, New-York.*

Let the equation of the given ellipse be

$$a^2y^2+b^2x^2=a^2b^2 \tag{1}$$

The equation of the tangent to the point $(x'y')$ of its perimeter will be

$$a^2y'y+b^2x'x=a^2b^2 \tag{2}$$

with the condition that

$$a^2y'^2+b^2x'^2=a^2b^2. \tag{3}$$

The equation of the perpendicular, let fall from the centre upon this tangent, will be

$$a^2y'x+b^2x'y=0, \tag{4}$$

and that of the ordinate to the point of contact

$$x-x'=0. \tag{5}$$

We shall obtain the locus of their intersection by eliminating x', y' between the equations (3), (4) and (5). The two last give

$$x'=x,\ y'=\frac{-b^2}{a^2}y$$

these values substituted in the first give

$$a^2x^2+b^2y^2=a^4 \tag{6}$$

which may be written

$$\left(\frac{a^2}{b^2}\right)^2 . \ x^2+a^2y^2=\left(\frac{a^2}{b^2}\right)a^2$$

which is the equation of a new ellipse, having its transverse axis $2a$ in common with the first, and whose conjugate is a third proportional to the axes $2a$ and $2b$ of the first. This gives rise to the following theorem, which is, I believe, new in Analytical Geometry.

"If upon the transverse axis of a semi-ellipse, considered as the conjugate, and upon the same side we describe another semi-ellipse, similar to the first, and erect an ordinate cutting them in two points; the straight line drawn from the centre to one of these points will be perpendicular to the tangent drawn at the other."

We might also demonstrate this theorem by Descriptive Geometry, by considering the ellipse as the orthogonal projection of a circle.

Cor. (1.) If in the equations (1) and (5), we change b into $b\sqrt{-1}$, they will become

$$a^2y^2-b^2x^2=a^2b^2\,;\ a^2x^2-b^2y^2=a^4,$$

which proves that the same property holds good for two hyperbolas, having the same transverse axis, and which is a mean proportional between their fictitious axes.

Cor. (2.) It is evident, upon consideration, that the same property will exist either in the case of two spheroids of revolution, or of two hyperboloids of revolution, with one sheet, having the same equator whose radius will be a mean proportional between the lengths of their principal diameters, perpendicular to the plane of this equator; this perpendicular will be intersected by the normal let fall from the centre upon the tangent plane, to a point in the other surface.

Second Solution.—*By Mr. O. Root, Vernon, New-York.*

Let $a^2y'^2+b^2x'^2=a^2b^2$ be the equation of the given ellipse, and xy the co-ordinates of the required locus.

By the question, $x'=x$.

And $\frac{b^2y'}{a^2x'}$ being the tangent of the angle (tx), $\frac{a^2y}{b^2}=y$;

these values of $y'x'$ substituted in the equation of the given ellipse, there results

$$b^2y^2+a^2x^2=a^4,$$

which is the equation of an ellipse concentrical with the given one, and whose semi-axes are $\frac{a^2}{b}$ and a.

A similar locus resulted from the investigations of Professor Strong; Messrs. Benjamin Peirce; C. Gill; Analyticus, N. J.; Analyticus, N. Y.; L'Inconnu, Cincinnati; and Omicron, N. C. Ed.

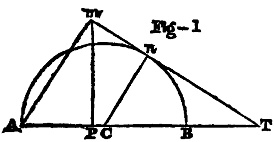
Question XV. (236.)—*By the same.*

It is required, to find the caustic by reflection, the reflecting curve being the locus of the intersections of a tangent to a given circle, with a line perpendicular to it, passing through a given point in the circle and the focus of incident rays being at the pole of the reflecting curve.

First Solution.—*By Mr. Samuel Ward, 3d, New-York.*

(1.) *Investigation of the required locus or reflecting curve.*

Let A (Fig. 1.) be the given point, which I shall assume as the pole of the reflecting curve and origin of rectangular co-ordinates. Let r=AC=An be the radius of the given circle, mT a tangent to any point n in its circumference, and Am a perpendicular thereto, will be the radius vector of the required curve, which we shall denote by z.



Let $\phi=m$AT$=n$CT be the variable angle made by z, upon the diameter AB, upon which line we shall reckon its successive values; by the evident similarity of the triangles TCn, TAm, we have the proportions.

$$\cos.\phi : r :: 1 : \text{CT}=\frac{r}{\cos.\phi} \therefore \text{AT}=\frac{r}{\cos.\phi}+r$$

and $$\text{CT} : \text{C}n :: \text{AT} : \text{A}m,$$

or $$\frac{r}{\cos.\phi} : r :: \frac{r}{\cos.\phi}+r : z,$$

whence, $$z=r(1+\cos.\phi), \quad (1)$$

which is the well known polar equation of the *cardioide*.

In order to refer it to rectangular axes, let AP$=x$, P$m=y$, (AB being the axis of x), and we evidently have

$$x^2+y^2=z^2, \text{ and } \cos.\phi=\frac{x}{z};$$

these values substituted in equation (1), give

$$(x^2+y^2)^{\frac{1}{2}} = r\left(1+\frac{x}{s}\right)$$

whence we obtain

$$y^2 = \left\{ \tfrac{1}{2}r \pm (rx+\tfrac{1}{4}r^2)^{\frac{1}{2}} \right\}^2 - x^2 \qquad (2)$$

for its rectangular equation.

As I have seen no notice taken in the "Diary" of this curious and beautiful curve, I shall take this opportunity of laying before its readers, some of its remarkable properties, &c.

History.—The *Cardioide* was first discovered by *Louis Carré*, a French mathematician of some celebrity, who flourished in the seventeenth and eighteenth centuries, and a paper of his may be found upon it in the *Mémoires de l'Academie*, 1705. It was also treated of by *Koersma*, in the *Philos. Trans.* 1741.

Genesis—The *Cardioide* may be generated by a point in the circumference of a circle, rolling upon an equal one, or in other words, it is an epycycloid, the radii of whose base and generating circle are equal.

Let Am (Fig. 2.) be any radius vector of the curve drawn through

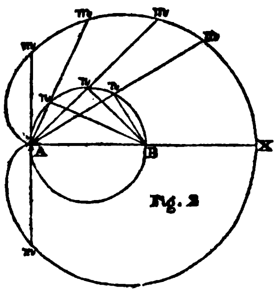


Fig. 2

any point n in the circumference of the base circle, then will the intercepted lines nm always $=$AB; for it is evident from the figure, that

$$An = 2r \cos.\phi,$$

whence, by (1) $\qquad Am - An = 2r\,;$

so that the curve may, by this property, be easily constructed, as in (Fig. 2.) by drawing nm always $=2r$; in form it closely resembles this figure.

Also.—If tangents be drawn through its cusp, they will always equal the axis. Whence, by the general formula, calling Ts the angle contained under the radius vector and the tangent, and differentiating equation (1)

$$ds = -r.\sin\phi.\,d\phi$$

we have

$$\tan. \mathrm{T}z = \frac{z}{r.\sin.\phi}.$$

The tangent at X is perpendicular to AX, and AX is a tangent to the curve at A, at which point it is evident that the curve becomes a *cusp* of the first kind. It may also be well to observe, that a tangent at the point m, the extremity of the perpendicular Am, will be inclined to that line in an angle of 45°.

Area.—To find this, let both sides of equation (1) be squared and multiplied by $d\phi$, whence, taking the integral between $\phi = o$ and $\phi = \pi$, and calling a one half the whole area, we have

$$a = 2r^2\pi + r^2\int \cos\phi . d\sin\phi ;$$

whence, as the last term is the expression for the area of the circle,

$$2a = 6r^2\pi,$$

or the whole area is six times the area of the base circle.

Length.—By general formula of roctification, we have

$$\int (z^2 d\phi^2 + dz^2)^{\frac{1}{2}} = 4r\int \cos\frac{\phi}{2} . d\phi ;$$

whence, by integrating between $\phi = o$ and $\phi = \frac{\pi}{2}$, and calling l one fourth the length, we have between these limits

$$l = 4r,$$

or the whole length is sixteen times that of the radius of the generating circle.

Evolute—From the nature of epycycloids, the evolute of the cardioide is another cardioide, the radius of whose base circle $= \frac{r}{3}$.

Involute.—Upon the same principle, we find that the involute of the cardioide is another cardioide, the radius of whose base circle $= 3r$.

Caustic.—The cardioide is the caustic by reflection of a circle, when the focus of incident rays is on the circumference of the circle, and the radius of its base $= \frac{1}{3}$ radius of given circle.

Natural property.—It is evident that the solid generated by the revolution of the curve around AX, will closely resemble an *apple*, every diametral section of which will be a cardioide.

Physical properties.—The cardioide possesses the following curious properties. 1. If a material point be supposed to move in it, attracted by a force directed to its cusp, the law of the force will be the inverse fourth power of the distance. 2. The velocity in it bears to the velocity in a circular orbit the invariable ratio $\sqrt{2} : \sqrt{3}$.

The *Cardioide* was so called by *Castilliani*, from its similitude to the figure of a *heart*.

(2.) *Investigation of the required caustic.*

Let $y=ax$—(3), be the equation of a ray of light referred to rectangular axes, and

$$y'^2 = \left\{ \tfrac{1}{2}r \pm (rx'+\tfrac{1}{4}r^2)^{\frac{1}{2}} \right\}^2 - x'^2, \tag{4}$$

that of the *cardioide*, or reflecting curve, x, y being the co-ordinates of any point in the caustic by reflection. We have for the differential of equation (4)

$$2y'dy' + 2x'dx' - \frac{\frac{1}{2}r^2 \pm r(rx'+\frac{1}{4}r^2)^{\frac{1}{2}}}{(rx'+\frac{1}{4}r^2)^{\frac{1}{2}}}dx' = o, \tag{5}$$

making

$$2y' = n; \text{ and } 2x' - \frac{\frac{1}{2}r^2 \pm r(rx'+\frac{1}{4}r^2)^{\frac{1}{2}}}{(rx'+\frac{1}{4}r^2)^{\frac{1}{2}}} = m, \tag{6}$$

(5) becomes

$$ndy' + mdx' = o. \tag{7}$$

Now, the equation of the reflected ray from the point x', y', of the reflecting curve, will be

$$y-y' = \frac{an^2+2mn-am^2}{m^2-2amn-n^2}(x-x'); \tag{8}$$

and we are enabled to pass from this ray to the next consecutive one by differentiating relatively to the variables, x', y' only. Whence we have

$$-dy' = \frac{an^2+2mn-am^2}{m^2-2amn-n^2}dx', \tag{9}$$ *

in which, by substituting for m and n, their values as given in (6), we obtain an equation, from which, if y' be eliminated by virtue of (4), and from this last and (8), eliminating x', we shall obtain the equation of the caustic required.

NOTE.—The caustic, or rather one of its points, might be easily determined from equation (1), by trigonometric considerations, since the incident and reflected rays make equal angles with the normal to any point in the reflecting curve.

SECOND SOLUTION.—*By Analyticus, New-Jersey.*

Let R=the radius of the given circle, D=the distance of the given point from its centre, r=the perpendicular from the given point to any tangent, ϕ=the angle made by r and D, then

$$r = \text{D} \cos. \phi + \text{R}, \tag{1}$$

is the polar equation of the reflecting curve, and it is evident that the focus of incident rays is at the origin of r.

Let r be any incident ray, r' the corresponding reflected ray, and

* (3) does not subject the ray to pass thro' the point (x'y') and (9) is not the differential of (8) with respect to x'y' since m and n are functions of x'y' and

2θ=their included angle, r''=the straight line joining the extremities of r and r', then (by trig.)

$$r''^2=r^2+r'^2-2rr' \cos. 2\theta \; (2), \text{ also } dr=-dr', \qquad (3)$$

because r, r' make equal angles with the reflecting curve at the point of reflection.

Again, because r'' remains invariable when r, r' are changed into their consecutive values, I take the differential of (2) considering r'' as constant, then by substituting the value of dr' as given by (2), and putting for $\frac{\sin. 2\theta}{1+\cos. 2\theta}$ its equal $\tan. \theta$, &c. I have

$$r'=\frac{r}{1+r.\frac{d(h \log. \cos. ^2\theta)}{dr}}. \qquad (4)$$

In this question

$$\cos. ^2\theta=\frac{r^2 d\phi^2}{dr^2+r^2d\phi^2}=\frac{r^2}{2Rr+D^2-R^2},$$

hence, and by (4) I have

$$r'=\frac{r(2Rr+D^2-R^2)}{4Rr+3D^2-3R^2},$$

whence the caustic is easily constructed.

Cor. Let p=the perpendicular from the origin of r to the tangent, to the reflecting curve at the point of incidence, then

$$\cos. \theta=\frac{p}{r},$$

and (4) is changed to

$$r'=\frac{r}{\frac{2dp}{dr}\times\frac{r}{p}-1}. \qquad (5)$$

Let R'=the radius of curvature of the reflecting curve at the point of incidence, then

$$\frac{dp}{dr}=\frac{r}{R'}$$

which changes (5) to

$$r'=\frac{r}{\frac{2r^2}{pR'}-1}. \qquad (6)$$

(4), (5) and (6) will serve to find the caustic, whatever the reflecting curve may be; if the reflecting curve is a circular arc, and the incident rays cut it at right angles very nearly, then R'=the radius of the

[illegible]

arc, and $\frac{r}{p}=1$ very nearly, hence by (6)

$$\left(r'-\frac{R'}{2}\right)\times\left(r-\frac{R'}{2}\right)=\frac{R'^2}{4} \qquad (7),$$

which result is well known.

Third Solution.—*By Mr. Benjamin Peirce, junr. Cambridge, Mass.*

Let the given point be the origin of co-ordinates, and the corresponding diameter of the circle the axis of the curve.

R = radius of the given circle.
r = radius vector of the reflecting curve.
r' = that of the caustic.
w = angle r makes with the axis.
w' = angle r' makes with it.
v = angle r makes with the reflecting curve.

$$r=R+R\cos. w.$$

$$\tan. v=\frac{rdw}{dr}=-\frac{1+\cos. w}{\sin. w}=\tan. \tfrac{1}{2}w,\ v=\tfrac{1}{2}w,$$

$180^\circ-2v=w=$angle r makes with the line described by the ray of light after reflection.

But $w-w'=$angle of r and r',
hence, $2w-w'=$supplement of the third angle of the triangle,
and $r : r'=\sin. (2w-w') : \sin. w$;
therefore, $r \sin. w=r' \sin. (2w-w')$ (A)

By differentiating, observing that the values of r' and w' are the same for two consecutive values of r and w, and that $dr=-R\sin. w\, dw$:—$R \sin.^2 w+R\cos. w+R\cos.^2 w=R(\cos. 2w+\cos. w)=2r'\cos. (2w-w')$. (B)

* Eliminating w from (A) and (B), we get the equation sought.

The method of *L'Inconnu*, was also extremely ingenious. Ed.

Question XVI. (237.)—*By Mr. Gerardus B. Docharty, L. I.*

Required, the content of the least sphere that shall touch three paraboloids given in magnitude and position.

First Solution.—*By Professor Strong, New-Brunswick, N. J.*

I shall generalize this question by supposing that the sphere is to touch any three given surfaces, whose equations are

$$u=0\ (1),\quad u'=0\ (2),\quad u''=0.\ (3)$$

Imagine the sphere to be placed so as to touch the three surfaces, and let the capitals X, Y, Z denote the rectangular co-ordinates of its centre, (x, y, z), (x', y', z'), (x'', y'', z'') those of its points of contact

* $\left(\frac{r'}{R}\right)^{\frac{4}{3}}-\left(\frac{r'}{R}\right)^{\frac{1}{3}}\cos w'=\frac{3}{4}\sin^{\frac{2}{3}} w'$

with (1), (2), (3) respectively; these co-ordinates, having the same origin as those of the centre of the sphere, then I have

$$r=\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}, \qquad (4)$$

$$r=\sqrt{(X-x')^2+(Y-y')^2+(Z-z')^2}, \qquad (5)$$

$$r=\sqrt{(X-x'')^2+(Y-y'')^2+(Z-z'')^2}. \qquad (6)$$

Now, since r is the shortest line that can be drawn from the point (X, Y, Z) to (1), I have

$$\frac{dr}{dx}dx+\frac{dr}{dy}dy+\frac{dr}{dz}dz=0 \qquad (7)$$

and by (1)

$$\frac{du}{dx}dx+\frac{du}{dy}dy+\frac{du}{dz}dz=0 \qquad (8)$$

By multiplying (8) by the indeterminate m, adding the product to (7), then putting the co-efficients of dx, dy, dz each equal to zero, I have

$$\frac{dr}{dx}+m\frac{du}{dx}=0,\ \frac{dr}{dy}+m\frac{du}{dy}=0,\ \frac{dr}{dz}+m\frac{du}{dz}=0;$$

or by eliminating the indeterminate,

$$\frac{dr}{dx}\times\frac{du}{dy}-\frac{dr}{dy}\times\frac{du}{dx}=0,\ \frac{dr}{dx}\times\frac{du}{dz}-\frac{dr}{dz}\times\frac{du}{dx}=0 \qquad (9)$$

By substituting in (9) the values of the partial differential co-efficients as given by (1) and (4), there will arise two equations involving X, Y, Z, x, y, z and known quantities, which with (1) will give x, y, z respectively in terms of X, Y, Z and known quantities; hence, r as given by (4) will be exhibited in terms of X, Y, Z and known quantities; by proceeding in the same manner with (2) and (5) as has been done with (1) and (4), r, as given by (5), will be found in terms of X, Y, Z and known quantities; also, in the same way, r, as given by (6), will be found in terms of X, Y, Z and given quantities; put then these three values of r equal to each other, and there will arise two equations, by which X and Y will be found in terms of Z and known quantities, hence, r will be expressed in terms of Z and known quantities; then as the sphere is to be a minimum

$$r=\text{min.}\ \therefore\ \frac{dr}{dz}=0 \quad (10),$$

which will give Z in terms of known quantities, and hence X and Y become known also, and the sphere becomes known in magnitude and position.

Second Solution.—*By Mr. Samuel Ward, 3d, New-York.*

Let R denote the radius of the sphere, and a, β, δ, the co-ordinates of its centre.

(1.) Let $\quad V=o,\ V'=o,\ V''=o, \quad$ (F)

represent the equations of the surfaces of the sixth order, generated by the revolutions of the curves upon which a circle rad. R must roll, in order that its centre shall describe the three generating parabolas of the paraboloids under consideration. The equations of the former curves may be found in the solutions of *Dr. Adrain* and *Mr. Nulty*, to question 172, No. X. of the "Diary." Since we shall suppose the surfaces (F) to envelop the three paraboloids, the former will intersect by pairs in three curves of double curvature, the equations of whose projections upon the planes of xz and yz, may be obtained by the successive elimination of x and y; these curves will also intersect in a number of points, which will evidently be the centres of *all* the spheres that can touch the three paraboloids. If then the value of R be found in terms of either of the co-ordinates, and its differential made equal to zero, we shall have determined the *least* sphere.

(2.) The paraboloids being considered as surfaces of revolution of the second order, the radii of the sphere will coincide with the productions of the normals to the three points of their contact with the sphere. These normals, it is well known, will intersect the axes of revolution. By means of the equations of the former (as found by *M. Puissant*, in his excellent *Recueil de Propositions de Geometrie*, p. 291. without the aid of the diff. cal.) and those of the axes of the surfaces, the co-ordinates of the points of intersection are easily obtained.

Then find expressions for the distances of each of these points from the point α, β, δ, from each distance deduct the normal of its meridian curve, equate the remainders to zero, and we have two equations, by means of which we may eliminate one of the variables, and find one of the others a function of the third. So that by substitution, there results an expression for R containing but one variable. The differentiation of which will again give us the radius and co-ordinates of the centre of the *least sphere*.

(3.) Or, let the summit of the first paraboloid be at the origin of rect. co-ordinates, and its axis coincident with that of z. Its equation will then be

$$V = x^2 + y^2 - pz = 0, \qquad (1)$$

and the distance between the centre of the sphere and a point on its surface, will be expressed by

$$\sqrt{(\alpha-x)^2+(\beta-y)^2+(\delta-z)^2}. \qquad (2)$$

Let the equation of the second paraboloid be

$$V' = (x+d)^2 + y^2 - p'z = 0, \qquad (3)$$

and the distance from a point on this surface, and α, β, δ will be expressed by

$$\sqrt{(\alpha-x-d)^2+(\beta-y)^2+(\delta-z)^2}. \qquad (4)$$

Now, (2) and (4) must be equal, each being $=$R, and R being also a normal, it must be a maximum or a minimum, according as the curved surface is concave or convex to the centre, or according as the axes of revolution of (F) $= p+$R, or $p-$R; $p'+$R, &c. whence,

$$\frac{d\text{R}}{dx}dx + \frac{d\text{R}}{dy}dy + \frac{d\text{R}}{dz}dz = 0,$$

and from equation (1) we find

$$\frac{dV}{dx}dx+\frac{dV}{dy}dy+\frac{dV}{dz}dz=o.$$

If we add the former to t, times the latter, we have

$$\frac{tdV}{dx}+\frac{dR}{dx}=o,\ \frac{tdV}{dy}+\frac{dR}{dy}=o,\ \frac{tdV}{dz}+\frac{dR}{dz}=o;$$

transferring the second terms of these equations, and multiplying crosswise, there results

$$\frac{dV}{dy}\times\frac{dR}{dx}-\frac{dV}{dx}\times\frac{dR}{dy}=o,$$

and

$$\frac{dV}{dz}\times\frac{dR}{dz}-\frac{dV}{dx}\times\frac{dR}{dz}=o.$$

Equation (1) gives,

$$\frac{dV}{dx}=2x,\ \frac{dV}{dy}=2y,\ \frac{dV}{dz}=-p,$$

and (2)

$$\frac{dR}{dx}=-(\alpha-x),\frac{dR}{dy}=-(\beta-y),\frac{dR}{dz}=-(\delta-z).$$

These latter substituted in the two former will give us

$$-2y(\alpha-x)+2x(\beta-y)=o, \qquad (5)$$

$$p(\alpha-x)+2x(\delta-z)=o, \qquad (6)$$

from (5) we have

$$x=\frac{\alpha}{\beta}y,$$

and from (6)

$$z=\frac{\alpha(2\delta-p)y+\alpha\beta p}{2y};$$

these last being substituted in (1) and ordered, give

$$y^3-\frac{\alpha\beta^2p(2\delta-p)}{2(\alpha^2-\beta^2)}y=\frac{\alpha\beta^2p^2}{2(\alpha^2+\beta^2)},$$

a cubic equation which must have at least one real root, which may be found by Cardan's Theorem, denoting it by ϕ, it will be a function of α,β,δ, thence $y=\phi'$ and $z=\phi''$, which are also functions of the same quantities; these, by virtue of equation (2), give

$$R=\sqrt{(\alpha-\phi')^2+(\beta-\phi')^2+(\delta-\phi'')^2}. \qquad (7)$$

In regard to the second and third surfaces, we have

$$R=\sqrt{(\alpha-\phi_{,})^2+(\beta-\phi_{,,})^2+(\delta-\phi_{,,,})^2},$$

$$R=\sqrt{(\alpha-m)^2+(\beta-n)^2+(\delta-k)^2},$$

in which, $\phi_{,}$, $\phi_{,,}$, $\phi_{,,,}$, m, n, k, are functions of α, β, δ. Now equating the three last, and eliminating R, we have

$$(\alpha-\phi)^2+(\beta-\phi')^2+(\delta-\phi'')^2=(\alpha-\phi_{,})^2+(\beta-\phi_{,,})^2+(\delta-\phi_{,,,})^2,$$

and $(\alpha-\phi)^2+(\beta-\phi')^2+(\delta-\phi'')^2=(\alpha-m)^2+(\beta-n)^2+(\delta-k)^2;$

from these equations, find two of the variables α, β, δ, in terms of the third, and the constants they contain, and write these values in the value of R, and since this is to be a minimum, its differential co-efficient being made equal to zero, will give the value of one of the co-ordinates of the centre of the touching sphere, whence we obtain those of the other two, and consequently R=min. is again determined.

QUESTION XVII. (238.)—*By a Student of Columbia College, N. Y.*

It is required to investigate the nature and properties of the curve along which a circle must roll, in order that its centre shall generate a cycloid.

FIRST SOLUTION.—*By Mr. J. S. Van de Graaff, Lexington, Kentucky.*

Let the rectangular co-ordinates x, y, of the required curve, have their origin at the centre of the given cycloid; the ordinate y being perpendicular to the base. Put c, to denote the axis of the cycloid; and let r be the radius of the circle rolling upon the curve required. Let a tangent be drawn to that point of the curve corresponding with the co-ordinates x, y, and intersecting the base of the cycloid produced if necessary. From the same point draw a line perpendicular to the curve, or tangent, and make it $=r$. The extremity of this line is then a point in the cycloid by the nature of the question. Let u, v, be the corresponding co-ordinates of the cycloid, and put $w=v-y$.

The value of the sub-tang. being

$$\frac{ydx}{-dy}$$

we easily obtain the equation,

$$y^2+\frac{y^2dx^2}{dy^2}+r^2=(w+y)^2+\left(\frac{ydx}{-dy}-(r^2-w^2)^{\frac{1}{2}}\right)^2;$$

and this, when reduced, becomes,

$$\frac{w}{(r^2-w^2)^{\frac{1}{2}}}=\frac{dx}{-dy}.$$

By differentitaing the obvious equations, $u=x+(r^2-w^2)^{\frac{1}{2}}$, $v=w+y$, and substituting for

$$\frac{w}{(r^2-w^2)^{\frac{1}{2}}},$$

its value just obtained, the result is

$$\frac{du}{-dv}=\frac{dx}{-dy}.$$

But by the property of the cycloid

$$\frac{du}{-dv}=\left(\frac{v}{c-v}\right)^{\frac{1}{2}}$$

$y = 2R \sin^2 \frac{1}{2}\varphi - r \sin \frac{1}{2}\varphi$

$x = R(\pi - \varphi + \sin\varphi) - r \cos \frac{1}{2}\varphi$

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and consequently,

$$\frac{w}{(r^2-w^2)^{\frac{1}{2}}}=\left(\frac{v}{c-v}\right)^{\frac{1}{2}},$$

which gives

$$y=\frac{cw^2}{r^2}-w.$$

The sub-tang. is, therefore, evidently

$$=\frac{w}{(r^2-w^2)}^{\frac{1}{2}}\times\left\{\frac{cw^2}{r^2}-w;\right\}$$

and from this there is no difficulty in showing that tangents to the required curve and to the cycloid, corresponding to the co-ordinates x, y, and u, v, are parallel to each other; and hence, if at any point in the cycloid we draw a line perpendicular to the curve of the cycloid at that point, and make it $=r$, then the extremity of this line will be a point in the curve required.

If the equation, area, or length of the curve be required, suitable expressions for their values may be obtained, without difficulty, from the equations

$$\frac{w}{(r^2-w^2)}^{\frac{1}{2}}=\frac{dx}{-dy}, \text{ and } y=\frac{cw^2}{r^2}-w.$$

SECOND SOLUTION.—*By Professor Strong, New-Brunswick, N. J.*

Let r=the radius of the rolling circle, x, y the rectangular co-ordinates of its centre, x', y' those of its point of contact with the curve on which it rolls, these co-ordinates having the same origin, let $y=Fx$ (1) be the equation of the curve described by the centre, also

$$r^2=\sqrt{(x-x')^2+(y-y')^2}. \qquad (2)$$

Now, since the radius (r) drawn to the point of contact cuts the curve on which the circle rolls at right angles, I have by (2) considering x', y' alone as variable,

$$\frac{-dy'}{dx'}=\frac{x-x'}{y-y'}, \qquad (3)$$

and by taking the total differential of (2) I have

$$(dx-dx')\times(x-x')+(dy-dy')\times(y-y')=0,$$

which is reduced by (3) to

$$dx\times(x-x')+dy\times(y-y')=0,$$

or

$$\frac{-dy}{dx}=\frac{x-x'}{y-y'}, \qquad (4)$$

by (3) and (4)

$$\frac{dy'}{dx'}=\frac{dy}{dx};$$

hence tangents to the two curves at the extremities of r are parallel, as they evidently ought to be.

Put $$\frac{dFx}{dx}=F'x,$$

then, by (1) and (4)

$$F'x=\frac{x'-x}{Fx-y'} \qquad (5)$$

also by putting for y its value Fx in (2) I have,

$$r^2=\sqrt{(x-x')^2+(Fx-y')^2}, \qquad (6)$$

by (5) x becomes known in terms of x', y' and given quantities, then by substituting for x its value in (6) there will arise an equation in terms of x', y' and given quantities, which will be the equation of the curve on which the circle must roll.

Let $$y=R(\phi+\sin.\phi),\ x=R(1-\cos.\phi), \qquad (7)$$

then $$\frac{dy}{dx}=\frac{1+\cos.\phi}{\sin.\phi}=\cot.\frac{\phi}{2}=\frac{x'-x}{y-y'}=\frac{dy'}{dx'}, \qquad (8)$$

or $$y-y'=(x'-x)\ \tan.\frac{\phi}{2};$$

this substituted in (2) gives

$$x'-x=r\cos.\frac{\phi}{2},\ \text{or}\ x'=R(1-\cos.\phi)+r\cos.\frac{\phi}{2}, \qquad (9)$$

X

also, $$y-y'=r\sin.\frac{\phi}{2},\ \text{or}\ y'=R(\phi+\sin.\phi)-r\sin.\frac{\phi}{2}; \qquad (10)$$

(9) is easily changed to

$$x'=2R\sin^2\frac{\phi}{2}+r\cos.\frac{\phi}{2}, \qquad (11)$$

by (8)

$$\frac{dy'}{dx'}=\cot.\frac{\phi}{2}\ \therefore\ \sin^2\frac{\phi}{2}=\frac{dx'^2}{dy'^2+dx'^2}\ \text{and}\ \cos.\frac{\phi}{2}=\frac{dy'}{\sqrt{dx'^2+dy'^2}}$$

these values when substituted in (11) give

$$x'=\frac{2Rdx'^2}{dy'^2+dx'^2}+\frac{rdy'}{\sqrt{dy'^2+dx'^2}}; \qquad (12)$$

for the differential equation of the required curve, which is easily constructed by (9) and (10), and it is hardly necessary to observe that (7) are the equations of the cycloid, R being the radius of the generating circle, the co-ordinates being at the vertex, the axis of the cycloid being the axis of x.

THIRD SOLUTION.—*By L'Inconnu, Cincinnnti.*

1. Let the origin of co-ordinates be at the centre of the given cycloid; for I shall decompose the question into the following one—" Requir-

* $$y'=2R\cos^{-1}\frac{r\pm\sqrt{16R^2r^2-8Rx'}}{4R}-\frac{1}{8R}(r\mp\sqrt{\ })\cdot\sqrt{8Rx'-2r^2\mp\sqrt{\ }}$$

ed the nature and properties of the curve upon which a circle must roll, in order that its centre shall genetate *a given* cycloid."

2. Let (x, y) be the co-ordinates of the required curve, (x', y') those of the centre of the rolling circle and consequently of the cycloid, r the radius of the generatrix, r' the radius of the rolling circle. Then we shall have for the equations of the cycloid

$$y' = r'(a + \sin a),\ x' = r' - r' \cos. a,$$

and as

$$\frac{x - x'}{y' - y} = \frac{dy}{dx},$$

we have

$$y = r'(a + \sin a) - r\tfrac{1}{2} \cos. a, \qquad (1)$$
$$x = 2r' \sin {}^2\tfrac{1}{2}a + r \cos. \tfrac{1}{2}a, \qquad (2)$$

these two equations will give us the differential equation of the required curve

$$x = f(r'xy).$$

Its length and area may be determined from the two equations

$$\frac{x'}{\sqrt{(r''^2 + x'^2)}} = \frac{dx}{-dy} \qquad (3)$$

and

$$y = \frac{ax'^2}{r^2} - x'. \qquad (4)$$

Length $= 8r$—semi-rolling circle,

Its area

$$= \left(\frac{3r^3}{2} + \frac{r'^3}{4}\right) - 4rr' + \tfrac{1}{2} + \text{a series.}$$

Ingenious solutions were also sent to this interesting question, by Messrs. C. Gill; Benjamin Peirce; and Samuel Ward, 3d, its proposer. The equation found by Mr. Gill, was of an highly transcendental order; Mr. Peirce found the radius of curvature $= \left(\frac{ds'}{dt}\right)^2$ in which s' denotes the arc of the required curve, commencing with y, and t the distance which the circle has rolled since y was zero; the solution of Mr. Ward was very complete: he adopts the method of *Lacroix*, as given in his *Traité Complet du Calcul.* T. 1. c. IV. pp. 492—500. He observes, that "from a consideration of the question, it is evident that tangents to the cycloid and the required base are parallel to each other, and that for any point in the cycloid, the corresponding point in the required curve will be at the extremity of the radius, which is a normal to the former, and consequently perpendicular to the parallel tangent: *ergo*, a normal to the latter curve. From which nothing can be easier than to construct the curve." He then proceeds to find the equations of the normal and of the tangent to the two curves, by means of which he is enabled to deduce the equation, na-

ture, and remarkable properties of the required one. He concludes with the following remark. "The foregoing is a particular, though by no means easy case of the general problem of *Consecutive Intersections*, *Roulettes*, &c. first broached by John Bernoulli and Leibnitz, in their *Commercium Philosophicum*, (Lausannae et Genevæ sumptibus *Marci Michaelis Bousquet* et Sociorum, 1745.)" At the end of the chapter of Lacroix, referred to above, he remarks, 'that La Hire has proved synthetically in the *Mém. de L'Academie*, (1706), that *any curve being given, we may always find one, which, rolling upon another given curve, will engender the first by one of its points.*' I have examined the paper of M. La Hire, and can safely assert, that it can, in no way, be of assistance to the student of the present day, since it comes to no definite result, except that if of the *Roulette*, *Generatrix*, or *Base*,* any two be given, we may discover the third. M. Nicole, in the same journal, for (1707), enters into an analytical investigation of several particular cases, and lays down the principle, that all geometric curves which roll upon themselves, will engender by any points, either on or within their circumference other geometric curves, and the same with regard to mechanical curves. He commences with circles producing the epycycloids, and goes no farther than the cubical parabola. In each of his examples, he supposes each curve under consideration, to roll upon a similar curve, but does not vary the conditions of the problem. In the case of the cycloid, he supposes it to be generated by the rolling of a circle of a given radius, upon one whose radius is infinite, it consequently cannot be expressed in algebraic terms. A species of obscurity rests upon the extension of this beautiful branch of the theory of curves, and nothing would give me more pleasure than that some of our skilful analysts should enter into the full examination of it; the aspirant to mathematical knowledge would ever look upon such a favour with emotions of gratitude, and the indulgent and liberal editor of this publication would willingly give an insertion to such a paper."

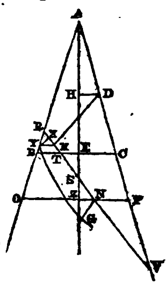
It is to be hoped, from the foregoing solutions and remarks, that such of our correspondents as supposed that the required curve was another cycloid, will be convinced that in no case, save that of a circle rolling upon a straight line or upon another circle, will its centre describe a concentric or similar curve. In the simple case of the parabola, the *Roulette* is of the *sixth* degree. Ed.

QUESTION XVIII. (239.)—*By Mr. Marcus Catlin, Elizabeth-town, N. J.*

In a given cone it is required to determine the greatest inscribed cone having its vertex in a given point in the slant surface.

* By *Roulette*, in the present question, we mean the cycloid; by *Generatrix*, the circle; and by *Base*, the required curve. Ed.

SOLUTION.*—*By Mr. Benjamin Peirce, Cambridge, Mass.*



In the figure annexed, ABC is the given cone; D, the given point; RNG, the plane of the base of the cone required, and $MT = My$ is the radius of the base.

Let then,

$$DM = r, \qquad HDM = w,$$
$$HAD = a, \qquad AE = h,$$
$$AH = z, \qquad TM = R,$$

we have then by the first solution to the succeeding question,

$$(r + \tan. a \sin. (w-a)z)^2 - 2 \sin. whr + (z-h)^2 = 0$$

$$R = -r \tan. w + \frac{z}{\cos. w} + \frac{h}{\cos. w}.$$

But as the required cone is a maximum

$$d.(R^2 r) = 2RrdR + R^2 dr = 0,$$
$$2rdR + Rdr = 0,$$
$$(3r \sin. w + z - h) \cos. w.dr + 2(r + z - h)rdw = 0.$$

Between this equation and the differential of the first one, eliminating dr and dw, we obtain a new equation between r and w, which, together with the first, is sufficient to determine their values.

Analyticus, N. Y.; Messrs. C. Gill; Omicron, N. C.; and O. Root, furnished ingenious methods of solving this question. ED.

QUESTION XIX. (240.)—*By the same.*

Required the locus of the centres of the bases of all the cones inscribed in a given cone having their vertices in a given point in the slant surface.

FIRST SOLUTION.—*By Mr. Benjamin Peirce, Cambridge, Mass.*

We shall make use of the same letters in this solution as in the preceding.

Let, moreover,

$$RV = 2A, \text{ and } RN = \tfrac{1}{2}RV = A,$$
$$NG = B, NT = \delta, Tx = z, xy = y,$$

we have then

$$DV = \frac{r}{\sin. (w-a)}, \qquad AD = \frac{z}{\cos. a},$$
$$AV = \frac{z}{\cos. a} + \frac{r}{\sin. (w-a)}$$

* The solution of Professor Strong to Questions XVIII. and XIX. will be found at Question XIX.

$$RV = 2A - \frac{2z \sin. a}{\sin. (w+a)} + \frac{r \sin. 2a}{\sin. (w-a) \sin. (w+a)},$$

$$PN = \frac{A \sin. (w-a)}{\cos. a},$$

$$ON = \frac{A \sin. (w+a)}{\cos. a},$$

$$B^2 = ON.PN = \frac{A^2 \sin. (w+a) \sin. (w-a)}{\cos.^2 a},$$

$$NM = MV - NV,$$

$$MV = r \cot. (w-a),$$

$$NM = \frac{r \sin. w \cos. w}{\sin. (w-a) \sin. (w+a)} - \frac{z \sin. a}{\sin. (w+a)} = R + \delta = \frac{A^2 - B^2}{A^2} z$$

$$z = \frac{A^2(R+\delta)}{A^2 - B^2},$$

$$y^2 = \frac{B^2}{A^2}(A^2 - z^2),$$

$$R^2 = y^2 + \frac{B^4}{A^4} z^2 = B^2 - \frac{B^2(R+\delta)^2}{A^2 - B^2},$$

$$A^2R^2 + zB^2\delta R + B^2\delta^2 = B^2(A^2 - B^2), \qquad (1)$$

$$\delta = \frac{LE}{\cos. w} = \frac{AL - h}{\cos. w} = \frac{z}{\cos. w} + \frac{r \cos. a}{\sin. a \sin. (w-a)} - \frac{h}{\cos. w},$$

$$R = NM - \delta = -r \tan. w - \frac{z-h}{\cos. w}.$$

Substituting the values of A, B, R, and δ in the equation (1), and reducing, we finally obtain for the equation sought

$$(r - \tan. a \sin. (w-a)z)^2 - 2 \sin. whr + (z - h)^2 = 0.$$

SECOND SOLUTION to QUESTIONS XVIII. and XIX.—*By Professor Strong, New-Brunswick, N. J.*

I suppose that the cones are right cones. Let n=the tangent of half the vertical angle of the given cone rad. (1), H, h=the perpendiculars from its vertex and the given point to its base, and the line of common section of the plane H, h, and the base be the axis of x; the axis of y being a perpendicular in the plane of the base through the foot of h, to the axis of x; h being the axis of z. Let a=the distance of the foot of h from the centre of the base, h'=the axis of any one of the inscribed cones, s=its slant height, r=the radius of its base, x, y, z the co-ordinates of the centre of its base, x', y', z' the co-ordinates of any point in the circumference of its base, x'', y'', z'' the co-ordinates of any point in the surface of the given cone.

Then $$n^2(H - z'')^2 = (x'' - a)^2 + y''^2, \qquad (1)$$

is the equation of the surface of the given cone; also,

$$(x-x')^2+(y-y')^2+(z-z')^2=r^2,\ (h-z)^2+y^2+x^2=h'^2,$$

$$(h-z')^2+y'^2+x'^2=s^2; \qquad (2)$$

since $h'^2+r^2=s^2$, by (2) I have

$$(h-z)z'=xx'+yy'+hz-(x^2+y^2+z^2). \qquad (3)$$

Now at the point where the base of the inscribed cone meets that of the given cone $z'=0$, $\therefore$ at this point the last of (2) and (3) become

$$h^2+y'^2+x'^2=s^2,\ xx'+yy'+hz-(x^2+y^2+z^2)=0, \qquad (4)$$

also because the circumference of the base of the inscribed cone touches the line of common section of its plane and the given base, I have, by taking the differentials of (4), considering x' and y' as alone variable

$$\frac{dy'}{dx'}=-\frac{x'}{y'}=-\frac{x}{y} \text{ or } y'=\frac{x'y}{x},$$

this value of y' when substituted in (4) gives (by denoting the value of x' by $x_{,}$),

$$h^2+\left(\frac{x_{,}}{x}\right)^2+(x^2+y^2)=s^2 \qquad (5)$$

$$hz+\left(\frac{x_{,}}{x}\right)\times(x^2+y^2)-(x^2+y^2+z^2)=0. \qquad (6)$$

Again, because the circumference of the inscribed base and the given surface have a common point, x'', y'', z'' become x', y', z' at this point, and (1) becomes

$$n^2(H-z')^2=(x'-a)^2+y'^2, \qquad (7)$$

also, because the circumference touches the line of common section of its plane and the given surface, at the same point, I have, by taking the differentials of the last of (2), of (3) and (7), considering x', y', z' as alone variable,

$$-(h-z')dz'\times x'dx'+y'dy'=0,$$

$$(h-z)dz'=xdx'+ydy',$$

$$-n^2(H-z')dz'=(x'-a)dx'+y'dy',$$

by the first and last of these

$$\frac{h-z'+n^2(H-z')}{a}=\frac{dx'}{dz'}$$

and by the first and second, I have

$$\frac{(h-z')y-(h-z)y'}{x'y-xy'}=\frac{dx'}{dz'},\ \therefore\ \frac{h-z'+n^2(H-z')}{a}=\frac{(h-z')y-(h-z)y'}{x'y-xy'}. \qquad (8)$$

x', y', z' can now be found by (3), (7) and (8), in terms of x, y, z and given quantities, then by substituting these values in the last of (2), s will be found in terms of x, y, z and known quautities, and by substituting this value of s, in (5) $\left(\frac{x_{,}}{x}\right)$ becomes known in terms of x, y, z and given quantities, and this value, when substituted in (6),

gives an equation in terms of x, y, z and given quantities which is the equation of the locus of the centres of the bases of all the inscribed cones. If the centres of the bases of the inscribed cones are supposed to be confined to the plane of H, h, then by putting $y=0$, I have

$$(x^2+z^2-hz-ax)^2\times(1+n^2)+\left((n^2H+h)x-a(h-z)^2\right)$$
$$=n^2(1+n^2)H^2x^2 \qquad (9)$$

for the equation of the locus in this case.

It is evident that the inscribed cone, when a maximum, has the centre of its base in the plane of H, h, hence, by putting $y=0$ in the aforesaid equations, r can easily be found in terms of x, z and known quantities; then since $h'r^2$=max. substitute for h', r their values in terms of x, z and known quantities, and eliminate x or z from the result by (9), then put the differential of $h'r^2$ thus found equal to zero, and there will arise an equation involving x or z and known quantities, whence the unknown quantity thus involved becomes known, and then the other unknown quantity is easily found; but as the process is attended with no difficulty except its length, I shall here leave the subject.

NOTE.—That the constants h and a are connected by the equation $a=n(H-h)$.

L'Inconnu found for the equation of the required locus, the expression

$$2\sin\phi\rho=r^2-2rh+h^2+(\rho-\tan a\sin(\phi-a)r^2);$$

which, as he remarks, evidently denotes a plane curve. The solutions of Analyticus, N. J. and Mr. Ward, N. Y., to the two last questions were effected by the analytical geometry of three dimensions.

ED.

QUESTION XX. (241.)—*By Professor Thomson, Nashville University, Tennessee.*

Required, the expression for the solidity of a solid of revolution, the equation of whose curve is $y=(\tan g.x)^2$, radius unity, the co-ordinates being taken on the concave side of the curve, and their origin at the centre of the generating circle.

FIRST SOLUTION.—*By L'Inconnu, Cincinnati.*

By taking the second differential co-efficient of the given equation, we have

$$\frac{d^2y}{dx^2}=\frac{2}{\cos.^2x}+\frac{6\sin^2x}{\cos.^4x}$$

this being positive, shows that as the convexity must be turned towards the axis of x, y is the axis of the solid.

Let $\sigma =$ the semi circumference of a circle, whose radius $=$ unity, then it is evident that

$$\int \sigma x^2 dy = 2\sigma' \int \frac{\sin x . x^2 dx}{\cos.^3 x} = \frac{\sigma x^2}{\cos.^2 x} - \sigma x.\tan.x + \int \sigma \tan.x.dx$$

$$= \sigma \left\{ \frac{x^2}{\cos.^2 x} - 2x \tan.x + l.\left(\frac{1}{\cos. x}\right)^2 \right\}$$

$=$the required solidity, supposing the integral to commence when $x = 0$, and it thus requires that no constant should be added.

The solidity, as found by Professor Strong, and Analyticus, N. J. does not materially differ from this. ED.

SECOND SOLUTION.—*By Mr. C. Gill, Sawpitts, N. Y.*

If the curve revolves on the axis on which x is taken, the content of the solid of revolution will be

$$= \pi \int y^2 dx = \pi \int dx \tan.^4 x = \pi(\tfrac{1}{3}\tan.^3 x - \tan.x + x),$$

between the circle, rad. y, and its vertex.

But if the revolution is made round an axis parallel to y, through the vertex, which I suppose is what the proposer means, its content will be

$$= \pi \int x^2 dy, = \pi x^2 y - 2\pi \int xy dx,$$

$$= \pi x^2 y - 2\pi \int x dx \tan^2 x, = \pi x^2 y - 2\pi \int x dx\ (\sec.^2 x - 1),$$

$$= \pi x^2 y + \pi x^2 (= \pi x^2 \sec.^2 x) - 2\pi \int x.d.(\tan. x),$$

$$= \pi x^2 \sec.^2 x - 2\pi x \tan.x + 2\pi \int dx \tan. x,$$

$$= \pi x^2 \sec.^2 x - 2\pi x \tan.x - 2\pi.h \log.\cos.x,$$

between the circle, radius x, and its vertex.

The solutions of Analyticus, N. Y., Messrs. Benjamin Peirce, and Samuel Ward, were similar to the above. Mr. Peirce found

$$\int \pi x^2 dy = \frac{\pi x^2}{\cos^2 x} - \pi x \tan.x - \pi \log.\cos x.$$

ED.

QUESTION XXI. (242.)—*By Mr. C. Gill, Sawpitts, New-York.*

A and B are given points in an indefinite straight line XY; from B* draw any right line BT, and describe a circle through A, having its

* This letter was incorrectly printed in the last number: (A.) ED.

centre in XY, which shall touch BT in T. To find the equation of the curve which is the locus of T; determine its area and length, and the surface and solidity of the solid generated by its revolution round XY.

FIRST SOLUTION.—*By the Proposer.*

Let AB $= a$, AT $= v$, and BAT $= \theta$. Then BOT $= 2\theta$ (O being the centre of the circle); ABT $= 90^\circ - 2\theta$, and ATB $= 90^\circ + \theta$. Hence,

$$v = a.\frac{\sin.(90^\circ - 2\theta)}{\sin.(90^\circ + \theta)} = a.\frac{\cos. 2\theta}{\cos.\theta},$$

is the polar equation.

When $\theta = 0^\circ$, $v = a$; when $\theta = 45^\circ$, $v = 0$, and when $\theta = 90^\circ$, $v = -\infty$; and then the ordinate is

$$= a \cos. 2\theta = -a;$$

therefore the curve has an asymptote perpendicular to XY at the distance a from A, but on the contrary side of it to B. Between $\theta = 90^\circ$ and $\theta = 180^\circ$, the curve is exactly the same as the former branch, but on the opposite side of the axis, forming at A what is termed a *punctum duplex*.

The rectangular equation is

$$x(x^2+y^2) = a(x^2-y^2), \text{ or } y = x\sqrt{\frac{a-x}{a+x}}$$

Many different constructions have been given to this beautiful curve. In the Gentleman's Diary, for 1783, it is described as being of Newton's forty-first species.

Quadrature.—Twice the area between AB, AT and the curve

$$= \int v^2 d\theta = a^2 \int d\theta.\frac{\cos.^2 2\theta}{\cos.^2\theta},$$

$$= a^2 \int d\theta(\sec.^2\theta - 4\sin.^2\theta),$$

$$= a^2(\tan.\theta + \sin. 2\theta - 2\theta)$$

Between $\theta = 0^\circ$ and $\theta = 45^\circ$, this gives the area of the loop

$$= a^2(2 - \tfrac{1}{2}\pi);$$

and between $\theta = 45^\circ$ and $\theta = 90^\circ$, the area between the infinite branches and their asymptote $= a^2(2 + \frac{1}{2}\pi)$.

Rectification.—The curve BT $= \int\sqrt{v^2 d\theta^2 + dv^2}$,

$$= a\int\frac{d\theta}{\cos.^2\theta}\sqrt{1+\sin.^2 2\theta},$$

$$= a\int\frac{d.2\theta}{1+\cos. 2\theta}\sqrt{1+\sin.^2 2\theta},$$

$$= a\left\{\int\frac{d2\theta}{\sin.^2 2\theta}\sqrt{1+\sin.^2 2\theta} - \int\frac{d\sin. 2\theta}{\sin.^2 2\theta}\sqrt{1+\sin.^2 2\theta}\right\}$$

$$= a \left\{ \sqrt{1+\text{cosec}^2 2\theta} - \text{hyp. log.} (\sin. 2\theta + \sqrt{1+\sin.^2 2\theta}) + \right.$$

$$\left. T - 2H - E \right\}.$$

Where H = arc of an equilateral hyperbola, whose semi-axes are unity, and abscissa

$$= \frac{\cos. (45^\circ - \theta)}{\sqrt{\sin. 2\theta}}$$

from the centre,

and tangent $\quad = T = \cos 2\theta \left(\frac{4 \sin.^2 2\theta}{\sin. 2\theta} \right)^{\frac{1}{2}}$

and E = an elliptical arc, whose semi-axes are $\sqrt{2}$ and 1, and abscissa 2 sin. $(45^\circ - \theta)$ from the centre. This, between $\theta = 0^\circ$ and $\theta = 45^\circ$, becomes

$$= \left\{ a\sqrt{2} - \log. (1+\sqrt{2}) + \right.$$

quadrantal arc of the ellipse — the excess of the two asymptotes of the hyperbola above the arc, both ways infinitely produced $\}$

$$= a \times 1,2447986,$$

twice which is the length of the whole loop. The other branch is, of course, infinite.

Surface.—The superficies of the solid

$$= 2\pi \int v \sin. \theta \sqrt{v^2 d\theta^2 + dv^2},$$

$$= 2a^2\pi \int \cos. 2\theta \tan. \theta. d\theta \sqrt{\frac{1+\sin.^2 2\theta,}{\cos.^2 \theta}}$$

$$= 2a^2\pi \int \frac{-\cos. 2\theta. d \cos. 2\theta}{(1+\cos. 2\theta)^2} \sqrt{2 - \cos.^2 2\theta},$$

$$= 2a^2\pi \left\{ \text{circ. arc, whose sine is } \sin.^2 2\theta - \frac{2+\cos. 2\theta}{1+\cos. 2\theta} \sqrt{1+\sin.^2 2\theta} \right\}$$

$$= (\text{between } \theta = 0^\circ \text{ and } \theta = 45^\circ),\ a^2\pi \left\{ \pi - 4\sqrt{2} + 3 \right\}$$

$= a^2\pi \times .4847382$, the surface of the loop. The surface of the other branch is infinite.

Cubature.—The solid

$$= \pi \int (v \sin. \theta.)^2 d(v \cos. \theta),$$

$$= \pi \int a^2 \cos.^2 2\theta \tan.^2 \theta. d(a \cos. 2\theta),$$

$$= a^3\pi \int d \cos. 2\theta \left\{ \frac{2}{1+\cos. 2\theta} - 2 + 2 \cos. 2\theta - \cos.^2 2\theta; \right\}$$

$$= a^3\pi \left\{ 2 \log. (1+\cos. 2\theta) - 2 \cos. 2\theta + \cos.^2 2\theta - \tfrac{1}{3} \cos.^3 2\theta) \right\}$$

Which gives the solidity of the solid formed by the loop

$$=a^3\pi\left(\log.4-\frac{4}{3}\right)=a^3\pi\times.05296103.$$

SECOND SOLUTION.—*By Mr. Samuel Ward, 3d, New-York.*

Let AB $=a$ (the figure will be readily supplied), TC $=r$, (C being the variable centre of the circle) BT $=z$, CBT $=\phi$.

The angle at T being a right angle, we have the proportion.

$$\sin\phi : r :: \cos\phi : z = r.\tan\phi$$

But BC $=a-r$, hence also $r=z.\tan\phi$; and by the square of the hypothenuse

$$z^2=a^2-2az.\tan\phi,$$

completing the square and reducing

$$z=a\,(-\tan\phi\pm\sec\phi) \qquad (1)$$

is the polar equation of the required locus.

Again, let x, y be the rectangular co-ordinates of T, origin at B; then since

$$r=\frac{a^2-ax}{2a\mp x} \text{ and } x^2+y^2=a^2\mp 2ar,$$

we easily find by substitution and reduction

$$x^2+y^2=\frac{\pm a^2x}{\pm 2a-x}, \qquad (2)$$

for its rectangular equation.

From (2) it is evident, that at A the curve will intersect in a *punctum duplex*, whose two branches will have an asymptote when $x=2a$. Its course may be easily traced from equation (1), and the length of the loop and surface of the solid may be thus obtained by Legendre's Tables of Elliptic Functions.

We here interrupt the solution of Mr. Ward, in order to give place to the similar results of our other correspondents. ED.

THIRD SOLUTION.—*By Analyticus, New-York.*

C being the centre of the circle, let TBC $=\phi$ and TB $=z$, then

$$z.\tan\phi=\text{TC}=\text{AC},$$

$$z.\sec\phi=\text{BC},$$

$$z.(\tan\phi+\sec\phi)=\text{AB}=a\ \therefore z=\frac{a.\cos\phi}{1+\sin\phi} \qquad (1)$$

is the polar equation of the curve.

Quadrature.—Let A $=$ the area reckoned from A, and A $=$ the whole area taken between $\phi = 0$, to $\phi = 90^{\circ}$, then

$$dA = \frac{a^2}{2}(2\sin\phi - 1 + \cos. 2\phi)d\phi,$$

$$A' = a^2\left(1 - \frac{\pi}{4}\right) = 0.2146 \times a^2,$$

Rectification.—Let $s =$ the length of the curve reckoned from A, and $s' =$ the whole length, then

$$ds = -d(1+\cos^2\phi)^{\frac{1}{2}} + \frac{a.d\cos\phi}{\sqrt{1+\cos^2\phi}} + \frac{a.\sqrt{2}.d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}$$

$$-a\sqrt{2}.d\phi \times \sqrt{1-\tfrac{1}{2}\sin^2\phi},$$

Integrating between the above limits, we have

$$s' = 1.24471 \times a \text{ nearly};$$

The integrals of the third and fourth terms of the value of ds, are found by Legendre's Tables of Elliptic Functions.*

Superficies.—Let S $=$ the surface of the solid reckoned from A, and $S' =$ the whole surface when taken between the same limits, then

$$dS = \pi a^2 \times \left(\frac{d\cos.^2\phi}{\sqrt{1+\cos.^2\phi}} \frac{d\cos.^2\phi}{\sqrt{1-\cos.^4\phi}} - d\left(\frac{\sin\phi\sqrt{1+\cos^2\phi}}{1+\sin\phi}\right)\right.$$

$$S' = \pi\frac{a^2}{2} \times (\pi + 3 - 4\sqrt{2}) = 0.76152 \times a^2.$$

Also obtained by Elliptical Functions.

Cubature.—Let $\Sigma =$ the solid reckoned from A, $\Sigma' =$ the whole solid taken between the same limits, we then have

$$d\Sigma = \pi a^3 \cos.\phi.\left(\frac{2}{1+\sin\phi} - 2 + 2\sin.\phi - \sin^2\phi\right)d\phi,$$

$$\therefore \Sigma' = \pi a^3 \times \left(2h.\log.2 - \frac{4}{3}\right) = 0.16638 \times a^3.$$

A similar method of investigation was pursued with equal success by Professor Strong and Mr. Benjamin Peirce.

L'Inconnu found by a similar method the *length* $= 1.245 \times a$, nearly; *area* $= 4a^2 - 2a^2\pi = 0.215 \times a^2$ nearly; *solidity* $= 0.1664 \times a^3$, nearly; *surface* of solid of revolution $= 0.762 \times a^2$, nearly. I would also remark, that in all of the foregoing solutions $\pi = 3.1415$, &c., nearly. Ed.

* It may be well to quote the above as a striking instance of the immense advantage afforded to the student by Legendre's "*Exercises de Calcul Integral,*" in which are contained his *Tables de Fonctions Elliptiques:* to find an integral by them is like finding a common logarithm in the Tables.

QUESTION XXII. (243.)—*By Mr. J. Leslie Payne, Halifax, Nova-Scotia.*

The upper extremity of a simple pendulum of given length is moved, according to a given law, along a given curve. Required the motions of the pendulum compatible with this restriction.

FIRST SOLUTION.—*By Professor Strong, New-Brunswick, N. J.*

I shall generalize the question, by supposing that two particles of matter M, m, are connected by an inflexible rod of a given length (a), and that m is compelled to move on a given curve, and acted upon in the directiou of the elements of the curve by a force which is denoted by F; supposing also that M, m, are subjected to the action of gravity.

Let then (x, y, z,) (x', y', z',) respectively denote the co-ordinates which define the positions of M, m, at any time t from the origin of the motion; these co-ordinates having the same origin, which is supposed to be in a horizontal plane passing through the place of M, when lowest; x being parallel to x', y to y', and z, z' vertical to the horizontal plane. Let (x', y', z') be connected by the equations

$$x'-fz'=0,\ y'-f'z'=0; \tag{1}$$

and put

$$\frac{dfz'}{dz'}=p,\ \frac{dfz'}{dz'}=q,\ ds=\sqrt{dx'^2+dy'^2+dz'^2}; \tag{2}$$

I also have

$$(x-x')^2+(y-y')^2+(z-z')^2=a^2,$$

which is satisfied by putting

$$x-x'=a\sin.\psi\cos.\phi,\ y-y'=a\sin.\psi\sin.\phi,\ z-z'=-a\cos.\psi \tag{3}$$

By the formula of Dynamics

$$\mathrm{M}\left(\frac{d^2x\delta x+d^2y\delta y+d^2z\delta z}{dt^2}+g\delta z\right)+m\left(\frac{d^2x'\delta x'+d^2y'\delta y'+d^2z'\delta z'}{dt^2}+g\delta z'\right)$$

$$-\mathrm{F}\left(\frac{dx'\delta x'+dy'\delta y'+dz'\delta z'}{ds}\right)+n(\delta x'-p\delta z')+n'(\delta y'-q\delta z')=0; \tag{4}$$

F being supposed to tend to increase the co-ordinates, and n, n' are two indeterminates.

Substituting the values of δx, δy, δz, as given by (3) in (4), then putting the co-efficients of $\delta x'$, $\delta y'$, $\delta z'$, $\delta\phi$, $\delta\psi$, each $=0$, there results

$$\frac{\mathrm{M}d^2x+md^2x'}{dt^2}-\mathrm{F}\frac{dx'}{ds}+n=0, \tag{5}$$

$$\frac{\mathrm{M}d^2y+md^2y'}{dt^2}-\mathrm{F}\frac{dy'}{ds}+n'=0, \tag{6}$$

$$\frac{\mathrm{M}d^2z+md^2z'}{dt^2}+(\mathrm{M}+m)g-\mathrm{F}\frac{dz'}{ds}-np-n'q=0, \tag{7}$$

$$d^2x \sin.\phi - d^2y \cos.\phi = 0, \tag{8}$$

$$\left(\frac{d^2z}{dt^2}+g\right)\sin.\psi+\frac{d^2x}{dt^2}\cos.\psi\cos.\phi+\frac{d^2y}{dt^2}\cos.\psi\sin.\phi=0; \tag{9}$$

these equations, together with (1), (2), (3) are sufficient to find the places of M, m, at any time, supposing F to be known at any time, together with the law of its variation.

The above equations comprehend a great variety of problems as particular cases, two or three of which as they are very simple, it may be well to notice. Let

$$z' = \text{const}, y' = 0, \phi = 0, \text{ then } y = 0;$$

also suppose that $\frac{d^2z}{dt^2}$ is very small when compared with g, and that ψ is so small that $\cos.\psi=1$, $\sin.\psi=\psi$, neglecting quantities of the second, &c. orders; hence, (6), (7), (8) do not exist, also n, n' are each $=0$, $ds=dx'$ and (5) becomes (by supposing m to be void of inertia,)

$$\frac{Md^2x}{dt^2} - F = 0;$$

put $F = MF'$, then

$$\frac{d^2x}{dt^2} = F', \tag{a}$$

also (9) becomes

$$\frac{d^2x}{dt^2} = -g\psi. \tag{b}$$

If F' = a small given quantity, then x is found by (a); and by (a) and (b), $\psi = \frac{-F'}{g}$ is known, $\therefore x'$ is easily found by $x-x'=a\psi=-\frac{aF'}{g}$. Also, if F' = a small (given) periodical function of t, then x is found by (a); and by (a) and (b) ψ becomes a known periodical function of t, hence x' is easily found.

Again, supposing that $F'=0$, and

$$\frac{dx'}{dt} = V = \text{a given velocity};$$

then (a) does not exist, but (by $x-x'=a\psi$,)

$$\frac{d^2x}{dt^2} = a\frac{d^2\psi}{dt^2};$$

hence, and by (b)

$$\frac{d^2\psi}{dt^2} = -\frac{g\psi}{a}, \text{ or } \frac{d\psi^2}{dt^2} = \frac{g}{a}\times\left(c-\psi^2\right), \text{ (}c\text{ the correction)};$$

let $\frac{d\psi'}{dt}$ = the initial value of $\frac{d\psi}{dt}$, then (since $\frac{dx}{dt}=0$ at the origin) by $x-x'=a\psi$, $\frac{d\psi'}{dt}=-\frac{V}{a}$; $\therefore c=\frac{V^2}{ag}$, and $\frac{d\psi^2}{dt^2}=\frac{g}{a}\times\left(\frac{V^2}{ag}-\psi^2\right)$; hence the places of M, m, are easily found at any time.

SECOND SOLUTION.—*By L'Inconnu, Cincinnati.*

I. (1.) I shall first consider the question in the following extremely simple case. Let it be proposed to determine the motion of a material point, acted upon by gravity, and suspended by an inflexible, inextensible wire, devoid of gravity, to a point compelled to move in an horizontal line from an initial state of rest, by an impulsive force, which communicates to it a known velocity.

(2.) The motion will evidently take place, in the vertical plane which passes through the direction of the initial velocity. Let the point of suspension and the pendulum be referred respectively to two rectangular axes x, y; the first being taken upon the horizontal straight line, described by the point of suspension; and the second in the direction of gravity, coincident with the initial position of the pendulum. Then let x, x', y' be the co-ordinates of these points at the end of the time t; by the principle of D'Alembert, the velocity lost by each point by virtue of their mutual connexion, will be, for the first,

$$-\frac{d^2x}{dt}\text{; and for the second, } -\frac{d^2x'}{dt},\ gdt-\frac{d^2y'}{dt}.$$

By the principle of virtual velocities, we have for the equation of equilibrium,

$$\frac{d^2x}{dt}\,\delta x+\frac{d^2x'}{dt}\,\delta x'+\left(\frac{d^2y'}{dt}-gdt\right)\delta y'=0; \qquad (a)$$

moreover, since the distance between the proposed points is constant, and = the length of the pendulum = l, we have

$$(x-x')^2+y'^2=l^2, \qquad (b)$$

which gives, $(x-x')(\delta x-\delta x')+y'\delta y'=0.$ $\qquad (b')$

Eliminating any one of the variations δx, $\delta x'$, $\delta y'$ between (a) and (b'), this last will resolve itself into the two following:

$$y'\frac{d^2x}{dt}-\left(\frac{d^2y'}{dt}-gdt\right)(x-x')=0; \qquad (c)$$

$$y'\frac{d^2x'}{dt}+\left(\frac{d^2y'}{dt}-gdt\right)(x-x')=0, \qquad (d)$$

which joined to (b), will complete the solution of this case of the problem, since equations (b), (c), and (d) comprise the four quantities x x', y', t, and we also know the trajectory of the first point.

(3.) Equation (c) added to (d), gives

$$\frac{d^2x}{dt}+\frac{d^2x'}{dt}=0,$$

of which the integral is, $x+x'=ct+c'$, $\qquad (d')$

but at the origin, $x=0$, $x'=0$, $t=0$, $\dfrac{dx'}{dt}=0$, $\dfrac{dx}{dt}=v$,

v denoting the given impulsive velocity of the point of suspension; thus (d') becomes

$$x + x' = vt. \qquad (e)$$

The difference of (d) and (c) being taken in order to obtain a new integral, we have

$$y'\left(\frac{d^2x}{dt} - \frac{d^2x'}{dt}\right) - 2\left(\frac{d^2y'}{dt} - gdt\right)(x - x') = 0 \qquad (c')$$

which may be reduced to an equation between two variables by a transformation of the co-ordinates.

(4.) Let ϕ = the angle made by the direction of the pendulum with the vertical passing through the point of suspension, x = the abscissa of this point, and we have by the transformation

$$y' = l.\ \cos.\ \phi\ (\mathrm{I}),\ x - x' = l.\ \sin.\ \phi\ (\mathrm{II});$$

substituting these values in (c') and integrating, it will become

$$l(1 + \sin.\ ^2\phi)d\phi^2 - 4g.\ \cos.\ \phi dt^2 = cdt^2.$$

Let us next determine c. We evidently have from (e) and (II) the following equations

$$2x = vt + l.\ \sin.\ \phi\ ;\ 2x' = vt - l.\ \sin.\ \phi\ ;\ 2\frac{dx'}{dt} = v - l\ \cos.\ \phi\ \frac{d\phi}{dt},$$

from these there results at the origin

$$\phi = 0,\ \frac{d\phi}{dt} = \frac{v}{l};$$

whence we have

$$c = \frac{v^2 - 4gl}{l}, \qquad (\mathrm{III})$$

we then obtain in place of (b), (c) and (d) the following system of co-ordinates:

$$\begin{aligned} y' &= l\ \cos.\ \phi, & (a')\\ 2x' &= vt - l\ \sin.\ \phi, & (b'')\\ 2x &= vt + l.\ \sin.\ \phi, & (c'')\\ l^2(1 + \sin.\ ^2\phi)\ d\phi^2 &= (v^2 - 4gl + 4gl.\ \cos.\ \phi)dt.^2. & (d'') \end{aligned}$$

Eliminating t between (b'') and (d''), we obtain for the differential equation of the trajectory of the inferior point referred to the axes x' and ϕ.

$$dx' = \frac{vl\sqrt{1 + \sin.\ ^2\phi}}{\sqrt{4v^2 - 16gl + 16gl\ \cos.\ \phi}}\, d\phi - \frac{l.\ \cos.\ \phi d\phi}{2}, \qquad (e')$$

from which we may draw many important conclusions prior to its integration.

(5.) Let Δ denote the smallest value of v^2, and let us investigate both its maxima and minima values, by following the changes of the quantities ϕ, x', x, when $v^2 = 8gl$, $> 4gl$, $= 4gl$, $< 4gl$ $\quad (f)$.

$\phi = 0$, when $v^2 \angle 8gl$ or $\angle 8gl$ $\therefore$ $\Delta = \pi, \angle\frac{\pi}{2}, = \frac{\pi}{2}, \angle \frac{\pi}{2}$; (f')

whence $$\frac{dx}{dt} = v, \frac{dx'}{dt} = 0, \frac{dy'}{dt} = 0. \qquad (v)$$

From which it is evident that the motion of the pendulum is periodic.

(6.) When $v^2 = 8gl$, ϕ increases to the value $\Delta = \pi$; this gives

$$\frac{d\phi}{dt} = 0, \frac{dx}{dt} = \frac{v}{2}, \frac{dx'}{dt} = \frac{v}{2}, \frac{dy'}{dt} = 0;$$

in this case the pendulum (which is vertical) is impelled in a direction contrary to that of gravity, and each of its extremities is animated with an equal horizontal velocity. It, however, can only attain this position at the end of an infinite period of time.

(7.) If ϕ begins to increase with t, we have

$$\frac{dx}{dt} = 0, \frac{dx'}{dt} = v, \frac{dy'}{dt} = 0:$$

so that the pendulum only varies from its original position, in that the impulse is communicated to the inferior point. In the negative region

$$\frac{dx}{dt} = v, \frac{dx'}{dt} = 0, \frac{dy'}{dt} = 0,$$

which show that the motion will *still* be *periodic*.

(8.) Let us now proceed to the integration of (e'). Let

$$z = \cos\ \phi;\ m = \frac{4gl}{v^2 - 4gl};$$

by a consideration of the cases in which $v^2 \angle 8gl$ $\therefore$ $m \angle 1$, it resolves itself into the form

$$\frac{dz}{(1+z)\sqrt{1-z}} \cdot \frac{1}{1-z} = \frac{dz}{(1-z)(1+z)\sqrt{1-z}},$$

whose integral (putting $\sqrt{1-z} = a$) as found by Lagrange, will be

$$-\frac{z}{1+v}\left\{\sqrt{\frac{v}{1-v}} \cdot \text{arc}\left(\tan. = a\sqrt{\frac{v}{1-v}}\right) - \frac{1}{2\sqrt{z}}\,\mathrm{L}\left(\frac{\sqrt{z}-a}{\sqrt{z}+a}\right)\right\} +$$

$$c = 0. \qquad (\mathrm{IV})$$

From which we obtain $t = 2c'$, for the interval of time taken up, in two successive passages across the vertical. The time of a period $= 4c'$, and in this time, the point of suspension describes a space $= 2vc'$; also the branches of the curve described from $\phi = 0$ to $\phi = \Delta$, and from $\phi = \Delta$ to $\phi = 0$, are respectively similar to those described from $\phi = -\Delta$ to $\phi = 0$, and vice versa. So that the trajectory itself is symmetrical with reference to this vertical.

(9.) Finally, for the positions of the pendulum, corresponding to points similarly placed on the similar branches of the trajectory, the

values of $\frac{dx}{dt}$, $\frac{dx'}{dt}$, $\frac{dy'}{dt}$, are found to be the same, and those of $\frac{d\phi}{dt}$ are equal, but with contrary signs.

II. (1.) The pendulum still being defined as in (I), let it be proposed to determine its motion when the point of suspension is acted upon by a force (F), compelling it to move according to a given law along a given curve.

(2.) I shall in this case refer it to three rectangular space axes; supposing the origin to be in an horizontal plane, passing through the position of the pendulum when lowest, and let $(x'y'z')$ $(x\ y\ z)$ denote the co-ordinates of the pendulum and point of suspension respectively; $(x'\ x)$ $(y'\ y)$ being parallel to each other, and $(z'\ z)$ vertical to the horizontal plane: moreover, let u, u', denote any two functions of z, l the length of the pendulum, s an arc of the curve, g the action of gravity, and t the time.

The Dynamical equations of condition, as in (I. § 2), become

$$\frac{d^2x'}{dt^2}\delta x' + \frac{d^2y'}{dt^2}\delta y' + \left(\frac{d^2z'}{dt^2} + g\right)\delta z' = 0, \qquad \text{(A)}$$

$$(x'-x)^2 + (y'-y)^2 + (z'-z)^2 = l^2, \qquad \text{(B)}$$

$$dx^2 + dy^2 + dz^2 = ds^2 \qquad \text{(C)}$$

$$x = u, \quad y = u' \qquad \text{(D)}$$

$$\frac{du}{dz} = f, \ \frac{du'}{dz'} = f'. \qquad \text{(E)}$$

(3.) Since (per question) we know the position of the point of suspension, its co-ordinates become known functions of t, therefore making

$$\left.\begin{aligned} x' - x &= \xi \\ y' - y &= \xi' \\ z' - z &= \xi'' \end{aligned}\right\}\text{(E)} \qquad \text{We have } \left.\begin{aligned} \delta x' &= \delta\xi \\ \delta y' &= \delta\xi' \\ \delta z' &= \delta\xi'' \end{aligned}\right\}\text{(G)}$$

$$\text{Put}\quad \left.\begin{aligned} \frac{d^2x}{dt^2} \qquad &= \mathrm{U}\ (\alpha) \\ \frac{d^2y}{dt^2} \qquad &= \mathrm{V}\ (\beta) \\ \frac{d^2z}{dt^2} + g &= \mathrm{W}\ (\gamma) \end{aligned}\right\}\text{(H)}$$

And I have

$$\left(\frac{d^2\xi}{dt^2} + \mathrm{U}\right)\delta\xi + \left(\frac{d^2\xi'}{dt^2} + \mathrm{V}\right)\delta\xi' + \left(\frac{d^2\xi''}{dt^2} + \mathrm{W}\right)\delta\xi'' = 0, \qquad \text{(I)}$$

for the general equation of the motion. From this it would appear that the pendulum moves with regard to the point of suspension as if that point were fixed, and as if the force F were applied to the pendulum in an opposite direction.

(4.) Equations (A), (B), (C), (D), (E), (I), will enable us in any case, to define the motion of the pendulum. Every simple case that can be investigated, will serve to confirm the conclusion which (I)

enable us to derive. Let F be supposed to produce uniform rectilinear motion, then

$$U=0,\ V=0,\ W=0, \qquad (A')$$

or if it be a projectile,

$$U=0,\ V=0,\ W=-g, \qquad (B')$$

which again shows that the pendulum moves as by the usual laws of its motion, viz. as if acted upon by a single force in the direction of g, and $=$ the diagonal of the parallelogram formed by F and g.

(5.) A transformation to polar co-ordinates will be suggested by Lagrange, (*Mec. Anal.* vol. ii. p. 194. §1. 15.) by which we obtain (s, s') still remaining vertical

$$\left.\begin{aligned} \xi &= l.\ \sin.\ \psi.\ \cos.\ \phi \\ \xi' &= l.\ \sin.\ \psi.\ \sin.\ \phi \\ \xi'' &= -\ l.\ \sin.\ \psi. \end{aligned}\right\}(\delta)$$

so that we now obtain for the complete solution of the problem, by equating to nought, the variational co-efficients and a substitution of their values as above, the equations

$$\frac{d^2x'}{dt^2}+U-F\,\frac{dx}{ds}+\lambda \qquad =0, \qquad (J)$$

$$\frac{d^2y'}{dt^2}+V-F\,\frac{dy}{ds}+\nu \qquad =0, \qquad (K)$$

$$\frac{d^2z'}{dt^2}+W-F\,\frac{dz}{ds}+\lambda f+\nu f' \qquad =0, \qquad (L)$$

$$d^2x'\sin.\ \phi-d^2y'\cos.\ \phi \qquad =0, \qquad (M)$$

$$\frac{d^2x'}{dt^2}\cos.\ \psi\cos.\ \phi+\frac{d^2y'}{dt^2}\cos.\ \psi\ \sin.\ \phi=0, \qquad (N)$$

$$\frac{d^2z'}{dt^2}\sin.\ \psi+g\sin.\ \psi \qquad =0. \qquad (O)$$

From which we can ascertain the position of the pendulum and point of suspension at any time, F being given as in the question.

III. (1.) An extremely general case of the problem (II) would be to suppose, the point of suspension to be material, and consequently that both are acted upon by gravity. Equation (I) would in this case change its form into one of five terms, having for co-efficients the respective masses of the two points, the force F and the indeterminates λ, ν. The subsequent equations (I to O) would be similarly affected.

(2.) I have attempted the extension of the problem to several particular cases, without much success in obtaining a neat result, among others, that of supposing the point of suspension to move on a curve surface, which was easily effected by the formulas of Lagrange, (*Mec. Anal.* vol. ii. § 2. 26. ;) also supposing the thread (l) *elastic* and *extensible*, which gives in our problem the general equation

$$\frac{d^2l}{dt^2}-\frac{l\sin.\ \psi^2\,d\phi^2+l d\psi^2}{dt^2}+F'+g\cos.\ \psi=0;$$

in which F' denotes (see *Mec. Anal.* ii. §1. 25.) the force with which the thread or wire tends to shorten itself.

(3.) If the point of suspension be supposed to move in any plane curve, the difficulty of obtaining a rigorous integral will increase with the order of the equation of the curve. In the conic sections an integral can only be obtained by great modification of the general conditions of the problem. The great obstacle will be found to consist in the necessary introduction of an indeterminate co-efficient to the angle made by the line drawn to $(x, y, z,)$ from the centre of the curve with the co-ordinate axis, with which its diameter is made to coincide. The question itself is highly ingenious, and admits of many beautiful, interesting, and useful varieties, but apologizing to the Editor of the Diary for the space which I have occupied, I here conclude the subject.

The solutions of Messrs. *Benjamin Peirce* and *Samuel Ward*, 3d, to the above question, were ingenious and perfectly general; both came in the general problem to the conclusion that the pendulum would move as if acted upon by a force = the resultant of gravity and the force which moves the point of suspension. Mr. Peirce, in a corollary, supposed the initial position of the pendulum to coincide with the axis of x, and the forces U, V, W, to be such as would cause it to perform a series of small oscillations round that axis; in which case, he observes, that its motion would be defined by the integral of Riccati. ED.

ACKNOWLEDGMENTS, &c.

The following gentlemen, favoured the Editor with solutions to the questions proposed in Article XXVIII. No. XII. The figures annexed to the names refer to the questions answered by each, as numbered in that article.

Professor *Theodore Strong*, Rutgers' College, New-Brunswick, N. J.; Mr. *Benjamin Peirce*, Mathematical Instructor in Harvard University; *L'Inconnu*, Cincinnati; and *Samuel Ward*, 3d, New-York, each solved ALL the questions.

Analyticus, New-Jersey, and Mr. *C. Gill*, Teacher of Mathematics, Sawpits, N. Y. solved all but 22: *Omicron*, North-Carolina, and Mr. *O. Root*, Vernon, N. Y. all but 15, 17, 22: Messrs. *James* and *Gerardus B. Docharty*, Long Island, all but 15, 17, 19, 22: Mr. *E. Loomis*, Baltimore, all but 15, 17, 18, 19, 20, 22: Messrs. *P. E. Miles*, Buffalo, N. Y. and *P Barton, junr.* Schenectady, N. Y. all but 15, 16, 17, 18, 19, 20, 21, 22: *Analyticus*, New-York, 13, 14, 15,

16, 17, 18, 19, 20, 21 : Messrs. *William Lenhart*, York, Penn. and *N. Vernon*, Fredericktown, Md. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 : *Nassau*, Long Island, 1, 2, 4, 6, 8, 12, 13, 14 : Mr. *Thomas Mooney, junr*. Brooklyn, L. I. 1, 2, 6, 10, 13, 14 : Mr. *John F. Jenkins*, Middleton Academy, 1, 2, 4, 6, 9, 12 : Mr. *Francis Sherry*, New-York, 9, 10 : Mr. *W. Hanson*, Lawrenceville, N. J. 1, 2, 3 : Mr. *William Vodges*, Philadelphia, 4 : and Mr. *J. S. Van de Graaff*, Lexington, Ky. 17.

Those gentlemen who have no acknowledgments for solutions to *their own questions*, have sent no solution.

It is also proper to observe, that Mr. *Benjamin Peirce* has, since the foregoing sheets were struck off, furnished me with less numbers which will satisfy the rationality of x and y, in Question V. than those stated in the solution of its ingenious proposer, Mr. *William Lenhart*. They are as follows,

$$a = 5,\ b = 63 \text{ which make } x = 6,\ y = \tfrac{2}{3}.$$

Ed.

ARTICLE XXXI.*

Mr. Editor,

In reply to the 24th article of the last Diary, (No. XI.) I observe that in my solution I supposed the terms there given ought to be rejected: for the question required "that the initial angle made by the plate and vertical should be small, and that the friction should be just sufficient to make the plate's circumference tend to roll without sliding." I hence inferred that the plate was to roll with the minimum friction, (or with the least restriction,) and that the oscillations were to be small, and I endeavoured to solve the problem accordingly. I will now add the terms mentioned (multiplied by M, the mass of the plate,) to the developement given at the 49th page of the 10th Diary; then by putting the co-efficients of $\delta\phi$, $\delta\psi$ each $= 0$, I have

$$cdp + \mathrm{MR}^2 \frac{d\theta}{dt} \sin.\theta d\psi = 0; \qquad (1)$$

$$d.(\mathrm{A} \sin.^2\theta \frac{d\psi}{dt} - cp \cos.\theta) - \mathrm{MR}^2 \frac{d\theta}{dt} \sin.\theta d\phi = 0. \qquad (2)$$

Now, I neglect the second terms of (1) and (2), because $\frac{d\theta}{dt}$ is to be a small periodical quantity, and because they are to be integrable with the least restriction; hence,

* This article, as will be seen by the date at the end of it, should have been inserted in No. XII. Ed.

$$cdp = 0, \text{ and } d\,(\mathrm{A} \sin.^2\theta \frac{d\psi}{dt} - cp \cos.\theta) = 0;$$

these equations are integrable independently of each other, and they agree with the results at the 49th page of the 10th Diary. By neglecting the second term of (1), and then integrating, I have $p =$ const. then writing dt under $d\phi$ in (2), and substituting p for $\frac{d\phi}{dt}$, it becomes integrable, and the results given at the 44th and 45th pages of the 10th Diary may be found; but this process restricts the integration of (2) to that of (1), which is virtually the restriction noticed at the 54th page of the same Diary.

I will now consider the question in another manner: to the end proposed I multiply (1) by $d\phi$, and (2) by $d\psi$, then by adding the products I have

$$d\phi \times cdp + d\psi \times d.(\mathrm{A} \sin.^2\theta \frac{d\psi}{dt} - cp \cos.\theta) = 0. \qquad (3)$$

I now observe that (3) is virtually the equation used at the 47th page of the 10th Diary, in the case of the variable projection; for by the equations there assumed

$$d\psi - \cos.\theta d\phi = 0, \text{ and } \delta\psi - \cos.\theta\delta\phi = 0,$$

hence, $$\frac{d\psi}{d\phi} = \frac{\delta\psi}{\delta\phi} \therefore d\psi\delta\phi - d\phi\delta\psi = 0,$$

which reduces the expression given in the 24th article to zero: this process is evidently the same as to change $\delta\phi$, $\delta\psi$ into $d\phi$, $d\psi$, which being done, the formula used at the place cited becomes the same as (3.) Indeed, since

$$d\psi = \cos.\theta d\phi \text{ and } \therefore \frac{d\psi}{dt} = \frac{p \cos.\theta}{\sin.^2\theta};$$

I have (by substituting these values in (3) and omitting the common factor $d\phi$,) the equation

$$cdp + (\mathrm{A} - \mathrm{C}) \times \cos.\theta d\,(p \cos.\theta) = 0,$$

which multiplied by p, integrated, &c. gives

$$(\mathrm{A} \cos.^2\theta + (\sin.^2\theta) \times p^2 = k^2,$$

which is the equation found at the bottom of the 47th page, for the r there used is the same as p, and $\mathrm{B}' = c$. Hence, similarly by substituting in (3)

$$\frac{d\psi}{dt} = \frac{cp(\cos.\theta - \cos\theta')}{\mathrm{A} \sin.^2\theta}, \qquad (4)$$

($\cos.\theta' =$ the initial value of $\cos.\theta$), and there results

$$(d\phi - d\psi \cos.\theta') \times cdp = 0,$$

or by rejecting $(d\phi - d\psi \cos.\theta') \times c$, I have

$$dp = 0, \text{ and } p = \text{const.} \qquad (5)$$

the equations (4) and (5) are the same as (3) and (1), found at the 50th and 49th pages of the 10th Diary.

Finally, I observe that (1) and (2) under their present forms are not integrable, but certain suppositions are to be made so as to render them integrable. One of these suppositions is, that $\frac{d\theta}{dt}$ is absolutely $= 0$, which gives $\theta =$ const. $\frac{d\phi}{dt} =$ const. $\frac{d\psi}{dt} =$ const. or $= 0$, (accordingly as its initial value $=$ a given quantity, or $= 0$,) and the plate describes a circle or straight line (accordingly as $\frac{d\psi}{dt} =$ a given quantity, or $= 0$,) on the horizontal plane with a constant inclination. Another supposition is that $\frac{d\theta}{dt}$ is to be a small periodical quantity, which has no sensible effect upon the motion of the plate, and in order to indicate these conditions the second terms of (1) and (2) are rejected; this is the supposition which I suppose from the statement of the question ought to be made. Another supposition is, that $\frac{d\psi}{dt} \times \frac{d\theta}{dt}$ sin. θ is to be rejected relatively to $\frac{dp}{dt}$, which is the supposition of the proposer; the term mentioned may be rejected either on account of the minuteness of $\frac{d\psi}{dt}$, of $\frac{d\theta}{dt}$, or of that of each; and does not therefore suppose (explicitly) that $\frac{d\theta}{dt}$ is to be small.

New-Brunswick, Nov. 2, 1830. T. STRONG.

ARTICLE XXXII.

Original Communications.

NO. I.*

GEOMETRICAL SOLUTION

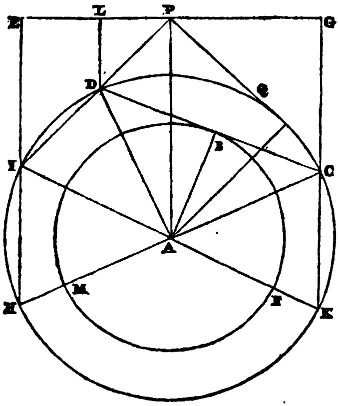
To the Prize Question proposed in No. V, and resolved in No. VI, by Professor Robert Adrain, LL.D.

* Circumstances have hitherto prevented the publication of this ingenious paper, which, however, arrived much too late for an insertion in the number for which it was intended.

Question XVIII. (86.) Or Prize Question.—*By Mr. J. H. Swale, Liverpool, England.*

A right line is given in position; to assign the position of a point in the periphery of a circle (given in position and magnitude) such, that thence drawing given and equal tangents, and demitting perpendiculars (upon the line given in position) from the extremities of the tangents; the rectangle of those perpendiculars shall be given, or a maximum or minimum.

Second Prize Solution—By Matthew Collins, Esq. Professor of Mathematics, in Limerick, Ireland.



Geometrical Analysis.—Suppose it done, and that B is the point required; in the periphery of the given circle, B, F, M, such that drawing the given and equal tangents BD, BC, and demitting the perpendiculars DL, CG, on the right line EG, given in position, their rectangle is a given space, a max. or a min. Take the centre A and join AD, AC, which are evidently equal and given, and from A with either describe the circle DCH, produce GC to meet this circle again in K, draw the diameters KI and CH, join HI, which shall be equal and parallel to KC, and thence EI equal to CG, join ID and produce it to meet EG in P. Now, when the rectangle CG × DL is given, IC × DL is so too, but the three sides of the triangle ACD being given, the ∠ACD is given, but this angle is equal to EIP, (22d, 3 and 13, 1); hence the triangles IEP, and DLP, are similar and given in species. Therefore, IC has to IP, and DL to DP, a given ratio; and when the rectangle IC × DL is given, the rectangle IP × PD will be given. From P, draw the tangent PQ, and let M be the side of a square equal to the given rectangle, then because of the similar triangles CAB, PIC, and PDL, we have IE : DL : : IP : PD, and IC × DL : IP × PD : : $DL^2 : PD^2 : : BC^2 : AC^2 : : M^2 : PQ^2$ and BC : AC : : M : PQ; hence, PQ is given, and therefore as the radius AQ is given, the secant AP is given; hence, this construction. From the centre A, apply the secant AP to the line EG, at P with PE, make the angle EPI = the given ∠BCA, and from the points D and I demit the perpendiculars DL, IE, produce the latter to meet again in H, draw the diameter HC and the perpendicular CG, so shall the rectangle CG × DL be = the given rectangle, and B the required point, the demonstration whereof is evi-

dent from the preceding analysis. As to the limits, it is plain, when the rectangle is a min. that PQ and AP will be so, and consequently AP perpendicular to EG, which determines the minimum, and much the same way the other limit is determined. The same construction holds as well as the analysis, *mutatis mutandis*, when the line EG cuts the given circle BFM, or DCH.

NO. II.

ON PERFECT NUMBERS.

By Benjamin Peirce, Esq. Mathematical Instructor in Harvard University, Cambridge, Mass.

Definition.—A Perfect Number is one that is equal to the sum of all its factors, unity being included.

Euler demonstrated that $(2^{n+1}-1)2^n$ is a perfect number, when $2^{n+1}-1$ is a prime number. But I have never seen it satisfactorily demonstrated, that this form includes all perfect numbers; Barlow's attempt is plainly defective. It is my object, in the present paper, to show that there can be no other perfect number included in the forms a^m, $a^m b^n$, $a^m b^n c^p$, where a, b, and c are prime numbers and greater than unity.

Notation.—Represent by $\Sigma^m a^i$ the sum of all the powers of a, from zero to the mth power inclusive, that is, let

$$\Sigma^m a^i = 1 + a + a^2 + \ldots \&c. \ldots + a^m.$$

Let *wh* stand for the word integer, or whole number.

First Form, a^m.

The sum of the factors of a^m is,

$$1 + a + \&c. \ldots + a^{m-1} a^i = \Sigma^{m-1} a^i.$$

But by the definition of a^m, this sum is equal to a^m,

or $$\Sigma^{m-1} a^i = a^m.$$

Every term of each member of this equation is divisible by a, except the term 1 Therefore, a cannot be greater than 1, and no perfect number can be included in this first form.

Second Form, $a^m b^n$.

The sum of all the factors of $a^m b^n$ is,

$$1 + a + b + a^2 + ab + b^2 + \&c. \ldots + a^{m-1}b^n + a^m b^{n-1},$$
$$= 1 + a + b + a^2 + ab + b^2 + \&c. \ldots + a^m b^n - a^m b^n,$$
$$= (1 + a + a^2 \ldots \&c. \ldots + a^m)(1 + b + b^2 \ldots + \&c. \ldots + b^n) - a^m b^n,$$
$$= \Sigma^m a^i \times \Sigma^m b^i - a^m b^n.$$

But since $a^m b^n$ is a perfect number, this sum must equal $a^m b^n$,

or $$\Sigma^m a^i \times \Sigma^n b^i - a^m b^n = a^m b^n,$$
$$\Sigma^m a^i \times \Sigma^n b^i = 2a^m b^n.$$

So that $\Sigma^m a^i$ or $\Sigma^n b^i$ must be an even number, and as $\Sigma^m a^i$ cannot be divisible by a, and $\Sigma^n b^i$ cannot be divisible by b. We may suppose

$$\Sigma^m a^i = 2b^n,$$

and $$\Sigma^n b^i = a^m.$$

Hence, $$\Sigma^n b^i - 1 = a^m - 1,$$

and $$\frac{a^m-1}{b} = \frac{\Sigma^n b^i - 1}{b} = wh.$$

But we had $$2b^n = \Sigma^m a^i = \frac{a^{m+1}-1}{a-1};$$

therefore, $$\frac{a^{m+1}-1}{b} = wh.$$

So that $$\frac{a^{m+1}-1}{b} - \frac{a(a^m-1)}{b} = \frac{a-1}{b} = wh. = p \text{ and } 70;$$

and $$a = pb+1.$$

Again, $$\Sigma^m a^i - 1 = 2b^n - 1,$$

and $$\frac{2b^n-1}{a} = \frac{\Sigma^m a^i - 1}{a} = wh.$$

But $$a^m = \Sigma^n b^i = \frac{b^{n+1}-1}{b-1};$$

so that $$\frac{b^{n+1}-1}{a} = wh.$$

or $$2.\left(\frac{b^{n+1}-1}{a}\right) - b.\left(\frac{2b^n-1}{a}\right) = \frac{b-2}{a} = \frac{b-2}{pb+1},$$

which is impossible, unless

$$b-2 = 0, \text{ or } b = 2.$$

This gives, $$\Sigma^n b^i = \frac{b^{n+1}-1}{b-1} = 2^{n+1}-1 = a^m,$$

and $$\Sigma^m a^i = 2b^n = 2^{n+1};$$

hence, $$\Sigma^m a^i - a^m = \Sigma^{m-1} a^i = 1.$$

So that $$m-1 = 0, \text{ or } m = 1,$$

and $$a = 2^{n+1}-1.$$

Therefore, $(2^{n+1}-1)2^n$ is a perfect number, when $2^{n+1}-1$ is a prime number. This is the form given by Euler, and it is evidently a perfect number, for the sum of its factors is

$$(1+2+2^2+ \ldots \&c. \ldots +2^n) + (1+2+ \ldots \&c. \ldots 2^{n-1})(2^{n+1}-1)$$
$$= (2^{n+1}-1)+(2^n-1)(2^{n+1}-1)$$
$$= 2^n(2^{n+1}-1) \text{ as required.}$$

THIRD FORM, $a^m b^n c^p$.

The sum of the factors of $a^m b^n c^p$, is found as in the former case to be

$$\Sigma^m a^i \times \Sigma^n b^i \times \Sigma^p c^i - a^m b^n c^p = a^m b^n c^p$$

or $$\Sigma^m a^i \times \Sigma^n b^i \times \Sigma^p c^i = 2a^m b^n c^p.$$

Whence we may suppose, as before,

$$\Sigma^m a^i = 2b^{n'}c^{p'}$$
$$\Sigma^n b^i = a^{m''}c^{p''}$$
$$\Sigma^p c^i = a^{m'''}b^{n'''},$$

m'', m''', n', n''', p' and p'' being integers, or equal to zero. And satisfying the equations,

$$m'' + m''' = m,$$
$$n' + n''' = n,$$
$$p' + p'' = p;$$

it follows from $\Sigma^m a^i = 2b^{n'}c^{p'}$, that m and a are both of them odd numbers.

Case First.

Suppose $b > 2$ and $c > 2$. Then they must both be odd numbers, and n and p must both be even numbers.

1. *m cannot be greater than* 1. For suppose $m > 1$, then

$$\Sigma^m a^i = \frac{a^{m+1}-1}{a-1} = \left(a^{\frac{m+1}{2}} + 1\right)\left(\frac{a^{\frac{m+1}{2}}-1}{a-1}\right) = 2b^{n'}c^{p'}$$

Now $\dfrac{a^{\frac{m+1}{2}}+1}{b}$ or $\dfrac{a^{\frac{m+1}{2}}+1}{c}$ is an integer, we may suppose the former.

Then $\dfrac{a^{\frac{m+1}{2}}+1}{b} = wh.$ and $\dfrac{(a^{\frac{m+1}{2}}+1)-2}{b} = \dfrac{a^{\frac{m+1}{2}}-1}{b} = wh.$ is impossible.

We must therefore have

$$\frac{a^{\frac{m+1}{2}}-1}{c} = wh.$$

and $\dfrac{(a^{\frac{m+1}{2}}-1)+2}{c} = \dfrac{a^{\frac{m+1}{2}}+1}{c} = wh.$ is impossible.

Moreover $\dfrac{a^{\frac{m+1}{2}}-1}{a-1} = wh.$ and $a^{\frac{m+1}{2}}+1$ is an even number; we must therefore have

$$a^{\frac{m+1}{2}}+1 = 2b^{n'},$$

and $\dfrac{a^{\frac{m+1}{2}}-1}{a-1} = c^{p'} = \Sigma^{\frac{m+1}{2}-1} a^i.$

From this last it is plain that $\frac{m+1}{2}-1$ is an even number, and therefore, $\frac{m+1}{2}$ is an odd number, and $a+1$ is a factor of $a^{\frac{m+1}{2}}+1=2b^{n'}$.

Therefore, $a+1=2b^{d}$; d being any positive integer,
and $a=2b^{d}-1$,

$$2b^{n'}=(2b^{d}-1)^{\frac{m+1}{2}}+1.$$

The term containing b^{d} is $2\frac{m+1}{2}.\,b^{d}$, which must be divisible by $2b^{d+1}$, unless $d=n'$, which cannot be, since it would give

$$2b^{n'}-1=2b^{d}-1=(2b^{d}-1)^{\frac{m+1}{2}},$$

$$\frac{m+1}{2}=1,\ m=1, \text{ contrary to the present hypothesis.}$$

We must then have $\frac{m+1}{2}=eb^{g}$, e being prime to b, and g an integer.

Hence $\quad 2b^{n'}=(2b^{d}-1)^{eb^{g}}+1$

therefore, $\quad (2b^{d}-1)^{b^{g}}+1$ being a factor of $(2b^{d}-1)^{eb^{g}}+1$,

we may write $(2b^{d}-1)^{b^{g}}+1=2b^{h}$;

or $\quad (2b^{d}-1)^{b^{g}}-2b^{h}+1=0,$

or, developing the last terms, and D being a known function of d,

$$D.b^{3d}-2b^{g}(b^{g}-1)b^{2d}+2b^{d}b^{g}-2b^{h}=0,$$

$$D.b^{3d}-2b^{2g+2d}+2b^{2d+g}+2b^{d+g}-2b^{h}=0,$$

which evidently requires that $d+g=h$, and this being substituted, the equation becomes

$$D.b^{3d}-2b^{2g+2d}+2b^{2d+g}=0,$$

or $\quad D'.b^{4d}+4b^{g}(b^{g}-1)\left(\frac{b^{g}-2}{3}\right)b^{3d}-2b^{2g+2d}+2b^{2d+g}=0,$

which can be satisfied only by making $b=3$, and then it becomes

$$D''\,3^{g+3}+4.3^{3g+3d-1}-4.3^{2g+3d}-2.3^{2g+2d}+8.3^{3d+g-1}+2.3^{2d+g}=0:$$

so that $3d+g-1=2d+g$ or $d=1$, and the equation becomes

$$D''.3^{g+3}+4.3^{3g+2}-4\,3^{2g+3}-2.3^{2g+2}+8.3^{g+2}=0,$$

which is plainly impossible. Therefore m cannot exceed unity.

This fact reduces the equation for case first to these

$$1+a=2b^{n'}c^{p'},$$
$$\Sigma^{n}b^{i}\ =ac^{p''},$$
$$\Sigma\,{}^{p}c^{i}\ =b^{n'''};$$

for $m''+m'''=m=1$. Therefore $m''=0$ or $m'''=0$, and it is a matter of indifference which is made zero.

These equations give

$$a = 2b^{n'}c^{p'} - 1,$$

$$\Sigma^n b^i = 2b^{n'}c^p - c^{p''},$$

$$\Sigma^p c^i = \frac{c^{p+1}-1}{c-1} = b^{n'''}.$$

Hence $$\frac{c^{p''}+1}{b} = wh. \text{ unless } n' = 0.$$

and $$\frac{c^{p+1}-1}{b} = wh.$$

Also $$\frac{c^{p''(p+1)}+1}{c^{p''}+1} \text{ being an integer, } \frac{c^{p''(p+1)}+1}{b} = wh.$$

and $$\frac{c^{p''(p+1)}-1}{c^{p+1}-1} \text{ being an integer, } \frac{c^{p''(p+1)}-1}{b} = wh.$$

so that $$\frac{c^{p''(p+1)}+1}{b} + \frac{c^{p''(p+1)}-1}{b} = \frac{2}{b} = wh. \text{ which is impossible}$$

Therefore, $$n' = 0,$$

and $$a = 2c^{p'} - 1,$$

$$\Sigma.^n b^i = 2c^p - c^{p''} = a.c^{p''} = \frac{b^{n+1}-1}{b-1}$$

$$\Sigma^p c^i = b^n.$$

Hence, $$b^{n+1} - 1 = ac^{p''}(b-1) = b\Sigma.^p c^i - 1,$$

$$b = \frac{ac^{p''}-1}{ac^{p''}+\Sigma.^p c^i} = \frac{ac^{p''}-1}{A} = wh.$$

where $$A = ac^{p''} - \Sigma.^p c^i.$$

Hence, $$(ac^{p''} - 1 - A) \div A = \frac{\Sigma^p.c^i - 1}{A} = c\left(\frac{\Sigma^{p-1}c^i}{A}\right) = wh.$$

or $$\frac{\Sigma^{p-1}.c^i}{A} = wh. \text{ and } \frac{c^p-1}{A} = wh.$$

Also $$\frac{ac^{p''}-1}{A} - \frac{c^p-1}{A} = \frac{ac^{p''}-c^p}{A} = c^{p''}\left(\frac{a-c^{p'}}{A}\right) = wh$$

or $$\frac{a-c^{p'}}{A} = \frac{c^{p'}-1}{A} = wh.$$

Let the greatest common divisor of p and p' be $p_{,}$; x and y can be found such $$xp - yp' = p_{,},$$

then $$\frac{c^{xp}-1}{A} = wh. \text{ and } \frac{c^{yp'}-1}{A} = wh.$$

therefore $$\frac{c^{xp}-1}{A} - \frac{c^{yp'}-1}{A} = \frac{c^{xp}-c^{yp'}}{A} = c^{yp'}\left(\frac{c^d-1}{A}\right) = wh.$$

and $$\frac{c^d-1}{A} = wh. = \frac{c^d-1}{c^p - c^{p''} - \Sigma^{p-1}.c^i} = \frac{(c^d-1)(c-1)}{(c-1)(c^{p'}-1)c^{p''}-(c^p-1)}$$

or $$(c-1)\div\left((c-1)c^{p''}\frac{c^{p'}-1}{c^d-1}-\frac{c^{p-1}}{c^d-1}\right)=wh.$$

$= (c-1)\div(Pc-1)$, where P is an integer, unless $p''=0$,

which would give $\Sigma^n b^i = 2c^p-1$, or $\frac{2}{b} = wh.$ which is impossible.

Therefore, $\frac{c-1}{Pc-1} = wh.$ or $P=1$, for $P=0$ would make A and therefore b negative quantities.

Hence, $$(c-1)\,c^{p''}\left(\frac{c^{p'}-1}{c^d-1}\right)-\frac{c^p-1}{c^d-1}=c-1,$$

and by development

$$c^{p+1}-2c^p-c^{p''+1}+c^{p''}-c^{d+1}+c^d+c=0,$$

or $$c^p-2c^{p-1}-c^{p''}+c^{p''-1}-c^d+c^{d-1}+1=0,$$

which requires that $d=1$.

Then $c^p-2c^{p-1}-c^{p''}+c^{p''-1}-c+2=0;$

therefore, $p''=1,$

$$c^p-2c^{p-1}-2c+3=0.$$

Therefore $c=3$, for p being even cannot equal unity.

This gives $3^p-2.3^{p-1}-3=0,$

or $3^{p-1}-2.3^{p-2}-1=0.$

Hence $p=2$, and $a=2c^{p'}-1=5,$

$$\Sigma^p.c^i=1+3+3^2=13=b^n=b,$$

$$\Sigma^n.b^i=1+13=a\,c^{p''}=15, \text{ which is absurd.}$$

It follows from the preceding analysis, that b and c cannot be both different from 2. Let then

CASE SECOND, $b=2$.

The given equations become

$$\Sigma^m.a^i=2^{n'+1}c^{p'},$$
$$\Sigma^n.b^i=2^{n+1}-1=a^{m''}c^{p''},$$
$$\Sigma^p.c^i=2^{n'''}a^{m'''}.$$

m is necessarily odd.

$$2^{n'+1}c^{p'}=\frac{a^{m+1}-1}{a-1}=\frac{a^{\frac{m+1}{2}}-1}{a-1}\left(a^{\frac{m+1}{2}}+1\right).$$

1. Let $\frac{m+1}{2}$ be even.

Then $$\frac{a^{\frac{m+1}{2}}-1}{2^2}=\left(\frac{a^{\frac{m+1}{4}}+1}{2}\right)\left(\frac{a^{\frac{m+1}{4}}-1}{2}\right)=wh.$$

therefore, $\dfrac{a^{\frac{m+1}{2}}-1}{4}+\frac{1}{2}=\dfrac{a^{\frac{m+1}{2}}+1}{4}=wh.$ is impossible.

So that $\dfrac{a^{\frac{m+1}{2}}+1}{c}=wh.$

and $\dfrac{a^{\frac{m+1}{2}}+1}{c}-\dfrac{2}{c}=\dfrac{a^{\frac{m+1}{2}}-1}{c}=wh.$ is impossible.

Therefore, $a^{\frac{m+1}{2}}+1=2c^{p'}$,

and $\dfrac{a^{\frac{m+1}{2}}-1}{a-1}=2^{n'}=\left(a^{\frac{m+1}{4}}+1\right)\left(\dfrac{a^{\frac{m+1}{4}}-1}{a-1}\right).$

Hence, $a^{\frac{m+1}{4}}+1=2^{d}$,

and $a^{\frac{m+1}{4}}-1=2^{d}-2=2^{n'-d}(a-1)$,

or $n'-d=0,\quad \frac{m+1}{4}=1,\quad m=3.$

$2^{n'}=a+1,\ a=2^{n'}-1.$

1a. Let $n''' > 0$. p must be odd.

And $2^{n'''}a^{m'''}=\dfrac{c^{p+1}-1}{c-1}=\left(\dfrac{c^{\frac{p+1}{2}}-1}{c-1}\right)\left(c^{\frac{p+1}{2}}+1\right).$

1a' Let $\dfrac{p+1}{2}$ be even.

Then $\dfrac{c^{\frac{p+1}{2}}-1}{2^{2}}=\left(\dfrac{c^{\frac{p+1}{4}}+1}{2}\right):\left(\dfrac{c^{\frac{p+1}{4}}-1}{2}\right)=wh.$

Therefore $\dfrac{c^{\frac{p+1}{2}}-1}{c-1}=2^{n'''-1}=\left(c^{\frac{p+1}{4}}+1\right)\left(\dfrac{c^{\frac{p+1}{4}}-1}{c-1}\right)$

and $c^{\frac{p+1}{2}}+1=2a^{m'''}$.

Hence, $c^{\frac{p+1}{4}}+1=2^{e}$,

$c^{\frac{p+1}{4}}-1=2^{e}-2=2^{n'''-e-1}(c-1).$

Therefore, $e = n''' - 1$, $c = 2^{n'''-1} - 1$, $p = 3$.

But we had $c^{p'} = \dfrac{a^{\frac{m+1}{2}} + 1}{2} = 2^{2n'-1} - 2^{n'} + 1.$

Therefore, $(2^{n'''-1} - 1)^{p'} = 2^{2n'-1} - 2^{n'} + 1.$

If p' were odd, this would give $n''' - 1 = 1$, or $n' = 1$, both which are impossible. Therefore, p' is even, and being less than p or 3.

$$p' = 2.$$

Therefore, $2^{2n'''-2} - 2^{n'''-2} = 2^{2n'-1} - 2^{n'}.$

Hence, $\quad n' = n''' - 2.$

Therefore, $2^{2n'''-2} = 2^{2n'''-5}$, which is absurd.

Then $\quad \dfrac{p+1}{4}$ is not even.

$1a''$. Let $\dfrac{p+1}{2}$ be odd and greater than 1.

In this case, $\dfrac{c^{\frac{p+1}{2}} - 1}{c - 1} \div 2 = wh.$ is impossible,

and $\qquad \dfrac{c^{\frac{p+1}{2}} + 1}{c - 1} = a^{m'''}$, $c^{\frac{p+1}{2}} + 1 = 2^{n'''}$, unless $p = 1$,

but $\qquad \dfrac{c^{\frac{p+1}{2}} + 1}{c + 1} = wh.$

Therefore, $c + 1 = 2^{n_{,}}$, $c = 2^{n_{,}} - 1.$

Hence, $(2^{n_{,}} - 1)^{\frac{p+1}{2}} + 1 = 2^{n'''}$, impossible, unless $n_{,} = n'''$ and $p = 1$.

$1a'''$. Let $\dfrac{p+1}{2} = 1$, or $p = 1$, and $p' = 1$.

Then $\qquad c + 1 = 2^{n''} a^{m'''},$

$$c = 2^{n'''} a^{m'''} - 1,$$

$$2(2^{n'''} a^{m'''} - 1) = a^{\frac{m+1}{2}} + 1.$$

Here if $m''' > 0$, we have $\dfrac{3}{a} = wh.$ or $a = 3$,

$$a = 2^{n'} - 1 = 3,\ n' = 2,$$

$$c = \frac{a^2 + 1}{2} = 5,$$

$$c + 1 = 6 = 2^{n'''} a^{m'''},\ n''' = 1,\ m''' = 1,\ m'' = 2,$$

$$n = n' + n''' = 3,\ 3^{n+1} - 1 = 15 = a^{m''}c^{p''} = 3^2 = 9,$$

which is absurd.

But if $\quad m''' = 0$, we have $c = 2^{n'''} - 1$,

$$a^{m''} = a^m = 2^{n+1} - 1 = (2^{n'} - 1)^m = (2^{n'} - 1)^3.$$

Therefore, $n' = n + 1 = 1$, which is impossible.

Therefore, n''' cannot be greater than zero.

1*b*. Let $n''' = 0$, p must be even.

$$n' = n,\ 2^{n+1} - 1 = a^{m''}c^{p''},$$
$$= (2^n - 1)^{m''}c^{p''}.$$

Therefore, if $m'' > 0$, $\dfrac{2^{n+1} - 1}{2^n - 1} = wh.$ which is impossible, unless

$$n = 1, \text{ or } a = 1;$$

so that $\quad m'' = 0$, and $m''' = 3$,

$$2^{n+1} - 1 = c^{p''} = 2a + 1.$$

But $\quad c^{p'} = \dfrac{a^2 + 1}{2}.$

Therefore, $8c^{p'} = 4a^2 + 4 = c^{2p''} - 2c^{p''} + 5$;

therefore, $\quad \dfrac{5}{c} = wh.\ c = 5,$

$$8.5^{p'-1} = 5^{2p''-1} - 2.5^{p''-1} + 1.$$

If $\quad p'' = 1,\ 8.5^{p'-1} = 5 - 1 = 4$, which is impossible.

If $\quad p'' = 1,\ 7 = 5^{2p''-1} - 2.5^{p''-1}$, which is impossible;

therefore, $8.5^{p'-1} = 5^{2p''-1} - 2.5^{p''-1} + 1$ cannot be satisfied,

and $\quad \dfrac{m + 1}{2}$ cannot be even.

2. Let $\dfrac{m+1}{2}$ be odd and greater than unity.

Then $\quad \dfrac{a^{\frac{m+1}{2}} - 1}{a - 1} = c^{p'}$, where $p' > 0$,

and $\quad a^{\frac{m+1}{2}} + 1 = 2^{n'+1}.$

But $\quad \dfrac{a^{\frac{m+1}{2}} + 1}{a + 1} = wh.\quad$ Therefore, $a + 1 = 2^d$,

$$a = 2^d - 1, \text{ and } (2^d - 1)^{\frac{m+1}{2}} + 1 = 2^{n'+1},$$

Therefore, $d = n' + 1$, and $\dfrac{m + 1}{2} = 1$, $m = 1$, contrary to the present hypothesis.

3. Let $$\frac{m+1}{2} = 1, \text{ or } m = 1.$$

$$a + 1 = 2^{n'+1}c^{p'},$$
$$2^{n+1} - 1 = a^{m''}c^{p''},$$
$$\Sigma p.c^{i} = 2^{n'''}a^{m'''}.$$

3a. Let $n''' > 0$, p must be odd,

and $$2^{n'''}a^{m'''} = \frac{c^{p+1}-1}{c-1} = \left(c^{\frac{p+1}{2}} + 1\right)\left(\frac{c^{\frac{p+1}{2}}-1}{c-1}\right)$$

3a′. Let $\frac{p+1}{2}$ be even.

Then $$\frac{c^{\frac{p+1}{2}}-1}{c-1} = 2^{n'''-1}, \text{ and } c^{\frac{p+1}{2}} + 1 = 2a^{n'''} = 2a;$$

$$c^{\frac{p+1}{4}} + 1 = 2^{e},$$
$$c^{\frac{p+1}{4}} - 1 = 2^{e} - 2,$$
$$c^{\frac{p+1}{2}} - 1 = 2^{2e} - 2^{e+1} = 2^{n'''-1}(c-1).$$

Therefore, $e = n''' - 1$, and $\frac{p+1}{4} = 1$, $p = 3$,

$$2a = c^{2} + 1, c + 1 = 2^{n'''-1},$$
$$2a + 2 = 2c^{2} + 2^{2} = 2^{n'+1}c^{p'}. \text{ Therefore, } p' = 0,$$
$$2c^{2} + 2^{2} = 2^{n'+1},$$
$$c^{2} = 2^{n'} - 2, \text{ which is impossible.}$$

3a″. Let $\frac{p+1}{2}$ be odd and > 1.

Then, $$\frac{c^{\frac{p+1}{2}}-1}{c-1} = a^{m'''} = a,$$

and $$c^{\frac{p+1}{2}} + 1 = 2^{n'''}.$$

But $$\frac{c^{\frac{p+1}{2}}+1}{c+1} = wh. \text{ Therefore, } c + 1 = 2^{d};$$

$$(2^{d}-1)^{\frac{p+1}{2}} + 1 = 2^{n'''},\ n''' = d,\ \frac{p+1}{2} = 1, \text{ contrary}$$

to hypothesis.

$3a'''$. Let $\frac{p+1}{2} = 1$, or $p = 1$.

$$c + 1 = 2^{n'''}a^{m'''}.$$

$3a_{,}'''$. Let $m''' > 0$ or $= 1$, $m'' = 0$, $p'' = 1$, $p' = 0$.

$$c + 1 = 2^{n'''}a,$$
$$a + 1 = 2^{n'+1},$$
$$c = 2^{n+1} - 1 = 2^{n'''}a - 1,$$
$$a = \frac{2^{n+1}}{2^{n'''}},$$ which is impossible, since a is odd.

$3a_{,,}'''$. Let $m''' = 0$, $m'' = 1$.

$$a = 2^{n'+1}c^{p'} - 1 = \frac{2^{n+1} - 1}{c^{p''}},$$
$$2^{n'+1}c^{p} - c^{p''} = 2^{n+1} - 1 = 2^{n'+1}c - c^{p''}.$$

Therefore, $p'' = 1$.

And $$c = \frac{2^{n+1} - 1}{2^{n'+1} - 1};$$
$$a = 2^{n'+1} - 1,$$
$$2^{n+1} - 1 = ac,$$
$$c = 2^{n'''} - 1,$$
$$2^{n+1} - 1 = 2^{n+1} - 2^{n'+1} - 2^{n'''} + 1,$$
$$1 = 2^{n'-2} + 2^{n'''-1}.$$

Therefore, $n''' = 1$, and $c = 2 - 1 = 1$, which is impossible.

$3b$. Lastly, let $n''' = 0$, p must be even.

$$m''' = 1,\ m'' = 0,\ n' = n,$$
$$a + 1 = 2^{n+1}c^{p'},$$
$$2^{n+1} - 1 = c^{p''},$$
$$\Sigma^{p}.c^{i} = a = 2^{n+1}c^{p'} - 1.$$ Therefore, $p' = 0$.
$$a + 1 = 2^{n+1} = c^{p''} + 1,$$
$$a = c^{p''},$$ which is impossible.

Therefore there can be no perfect number of the form, $a^{m}b^{n}c^{p}$.

NO. III.

Diophantine Problem and Solution, No. X.

By William Lenhart, Esq. York, Pennsylvania.

PROBLEM.

To divide $n + 1$, into n, squares, each greater than unity: n being greater than 2.

SOLUTION.

Let $$\frac{v+x}{v-z},\ \frac{v+x'}{v-z},\ \frac{v+x''}{v-z},\ \&c.$$
represent the roots of the required squares; then, by the question,
$$\left(\frac{v+x}{v-z}\right)^2+\left(\frac{v+x'}{v-z}\right)^2+\left(\frac{v+x''}{v-z}\right)^2\ \&c. = n+1,$$
or, $nv^2+2v(x+x'+x''\ \&c.)+x^2+x'^2+x''^2\&c. = (n+1).(v-z)^2$.
Now, by assuming $(n+1).z^2 = x^2+x'^2+x''^2$ &c. we shall have
$$v = 2((n+1).z+x+x'+x''\ \&c.)$$ and hence the following

General Rule.—Divide $n+1$, times any square (z^2) into n, or $n-1$, squares having positive and negative roots, (no negative root, however, must be equal to, or greater than, the root (z) of the assumed square,) then to twice the difference between the sum of the positive and the sum of the negative roots, or if the roots are all positive, to twice their sum, add $2n+1$, times the root (z) of the assumed square, and you will have the common denominator of the required roots; and by adding to, and subtracting from, the sum of the denominator thus found and the root of the assumed square, the respective roots into which $(n+1).z^2$ is divided, according as they are positive or negative, you will have the roots of the respective numerators.

NOTE. I.—When the series of squares into which $(n+1).z^2$ is divided consists only of $n-1$, terms, the root of the deficient numerator will, in all cases, be the sum of the roots of the common denominator and of the assumed square (z^2).

Scholium.—It may readily be perceived, that as n and z increase, there will be some difficulty in dividing $(n+1).z^2$ into n, or $n-1$, squares, as the rule in the first instance requires. To obviate this, we shall point out a method which, it is presumed, will succeed in all cases. In the first place, take z, somewhat greater than $\frac{n+1}{3}$, and assume a series of squares whose roots are $+1, -1, +2, -2, +3, -3$, &c. to $2(z-1)$ terms, which series continue with squares whose roots are $z, z+1, z+2$, &c. until you have n, terms in all. Next take the difference between $(n+1).z^2$ and the sum of the assumed series of squares, which difference (D), if it is greater than the last square of the assumed series, subtract from the square next greater than itself, or if it is less, from the square next greater than the last square of the assumed series, and if the remainder (R) is not a square or the sum of two squares, proceed in the same manner with the next greater square, and so on, until you find a remainder (R) that is a square, or the sum of two squares included in the assumed series: then strike from the assumed series of squares, the square remainder (R), or two squares into which the remainder (R) is divided, and add to it the square from which the difference (D) was taken, and thus will $(n+1).z^2$ be divided into n, or $n-1$, squares prepared for the operation of the rule. It may be necessary sometimes to make the

assumed series of squares consist of $n+1$, terms, in which case the remainder (R) must be the sum of two or of three squares included in the assumed series, thence proceeding as above directed, we shall have another division of $(n+1).z^2$ into n, or $n-1$, squares to suit our purpose. We may also take $(n+2)$ terms, in which case the remainder (R) must be the sum of three or of four squares included in the assumed series, &c. It would be tedious to enumerate all the neat artifices that may be used in the preparation of a series of squares for the operation of the rule, I shall, therefore, leave them to the skill and ingenuity of those readers who may think proper to calculate the different cases, and proceed to illustrate what has been said in this scholium, and apply the rule to other and more complicated examples than those given by me in No. XI. of the Diary.

We here subjoin one of the numerous examples which Mr. Lenhart has furnished, as a practical illustration of his General Rule.

Example.—Divide 31 into 30 squares, each greater than unity.

Solution.—Here $n=30$, $n+1=31$, and $z \angle \frac{n+1}{3} = 11$, consequently $(n+1)z^2 = 31 \times 121 = 3751$. Assume a series of squares of 30 terms, as directed in the scholium, and its sum will be $= 3255$, which being subtracted from $(n+1).z^2 = 3751$, leaves $496 = D$. This difference (D) will be found to be greater than the last square of the assumed series of squares; we therefore commence our subtractions from the square next greater than 496, viz.

$$\left.\begin{array}{l}(23)^2-496=33\\(24)^2-496=80=(4)^2+(8)^2\\(25)^2-496=129\\(26)^2-496=180=(6)^2+(12)^2\end{array}\right\}\left\{\begin{array}{l}(27)^2-496=233=(8)^2+(13)^2\\(28)^2-496=288\\(29)^2-496=345\\(30)^2-496=404=(2)^2+(20)^2\end{array}\right\}$$

Now, each of the remainders that is the sum of two squares will furnish a suitable series of squares; but the remainder $233=(8)^2+(13)^2$ will produce the lowest denominator, because the difference between the root (27) of the square from which 496 is taken, and the sum of the roots of the squares into which the remainder $(R)=233$ is divided, is less than any of the others, viz.

$$\begin{aligned}27-(8+13)&=6,\\24-(4+8)&=12,\\26-(6+12)&=8,\\30-(2+20)&=8.\end{aligned}$$

This will appear evident by inspection, and from the nature of the operation. The two series of squares resulting from the remainders 180 and 404, will produce the same denominator, because $26-(6+12)=30-(2+20)$; but the numerators will be different. The roots of a series of squares resulting from the remainder $233=(8)^2+(13)^2$, and prepared for the operation of the rule will be $-1, +1, -2, +2, -3, +3, -4, +4, -5, +5, -6, +6, -7, +7, -$

8, -9, $+9$, -10, $+10$, 11, 12, 14, 15, 16, 17, 18, 19, 20, and 27. Hence, by the rule, $2 \times (216 - 55) + 61 \times 11 - 993 =$ root of common denominator, and the roots of the numerators will be 994, 995, 996, 997, 998, 999, 1000, 1001, 1002, 1003, 1004, 1005, 1006, 1007, 1008, 1009, 1010, 1011, 1013, 1014, 1015, 1016, 1018, 1019, 1020, 1021, 1022, 1023, 1024, and 1031. The sum of their squares is 30567519, which is equal to $31 \times (993)^2$.

NOTE II.—And in a manner precisely similar, we may divide $n - 1$, into n, squares, each less than unity, &c.; for changing the signs in the notation, we shall have

$$\left(\frac{v-x}{v+z}\right)^2 + \left(\frac{v-x'}{v+z}\right)^2 + \left(\frac{v-x''}{v+z}\right)^2 \text{ \&c.} = n - 1$$

which being reduced, and assuming $\overline{n-1}.z^2 = x^2 + x'^2 + x''^2$ &c. will give $v = 2\,(\overline{n-1}.z + x + x' + x''$, &c.) Hence, a general rule, as before. And by the aid of these two rules, we may readily solve the following questions, viz

Find n positive numbers, whose sum shall make unity, and such that each of the numbers be increased by unity, the respective sums shall be rational squares.

Find n positive numbers, whose sum, and also the sum of every $n - 1$ of them, may be rational squares.

NO. IV.

Diophantine Problem and Solution, No. XX.

By MICHAEL FLOY, Junr. Esq. New-York.

PROBLEM.

To make squares of the expressions,

$$a^2x^2 + a'x = \square,$$
$$b^2x^2 + b'x = \square,$$
$$c^2x^2 + c'x = \square.$$

SOLUTION.

If we divide by a^2, b^2, and c^2, and put for $\frac{a'}{a^2}$, $\frac{b'}{b^2}$, and $\frac{c'}{c^2}$, m, n, and p, the above equations become,

$$x^2 + mx = \square,$$
$$x^2 + nx = \square,$$
$$x^2 + px = \square.$$

To make $x^2 + mx$ a square, assume its root equal $x - f$, then

$$x^2 + mx = x^2 - 2fx + f^2. \qquad (1)$$

By substitution in the other two formulas, we have

$$\left.\begin{array}{l} f^2 + 2fn + mn = \square = A^2 \\ f^2 + 2fp + mp = \square = B^2 \end{array}\right\}$$

By subtraction

$$A^2 - B^2 = 2f(n - p) + m(n - p) = (2f + m)(n - p).$$

Now take $\quad A + B = 2f + m, \quad A - B = n - p,$

and we have $\quad A = f + \dfrac{m + n - p}{2};$

whence,

$$f^2 + 2fn + mn = (A^2) = f^2 + f(m + n - p) + \tfrac{1}{4}(m + n - p)^2,$$

$$\therefore f = \frac{mn - \frac{1}{4}(m + n - p)^2}{m - (n + p)};$$

And f being found, we have from (1),

$$x = \frac{f^2}{2f + m}. \qquad (2)$$

This general problem can be easily applied to many particular and interesting researches.

Applications.

Ex. 1.—To find three square numbers in geometrical proportion, such that each being increased by its square root, the sums may be squares.

App.—Let x^2, $4x^2$, and $16x^2$ denote the numbers.

Then $\quad x^2 + x,\ 4x^2 + 2x,$ and $16x^2 + 4x,$

or $\quad x^2 + x,\ x^2 + \frac{1}{2}x,\ x^2 + \frac{1}{4}x,$ must be squares.

Whence $\quad m = 1,\ n = \frac{1}{2},\ p = \frac{1}{4};$

And therefore,

$$f = \frac{7}{16},\ x = \frac{f^2}{2f + m} = \frac{49}{(14 + 16)16} = \frac{49}{480};$$

hence the numbers are $\left(\dfrac{49}{480}\right)^2, \left(\dfrac{98}{480}\right)^2, \left(\dfrac{196}{480}\right)^2.$

Ex. 2.—To find three squares in arithmetical progression, with the same conditions.

App.—Let x^2, $25x^2$, and $49x^2$ denote the numbers.

Then $\quad x^2 + x,\ x^2 + \frac{1}{5}x,$ and $x^2 + \frac{1}{7}x$ must be squares;

$$m = 1,\ n = \tfrac{1}{5},\ p = \tfrac{1}{7}:$$

hence, $f = -\dfrac{389}{3220},\ x = \dfrac{389^2}{2442 \times 3220} = \dfrac{151321}{7863240},$

and the numbers are, $\left(\dfrac{151321}{7863240}\right)^2 \left(\dfrac{756605}{7863240}\right)^2$ and $\left(\dfrac{1059247}{7863240}\right)^2.$

Ex. 3.—To find three squares in harmonic proportion, with the same conditions.

App.—Let $25x^2$, $49x^2$, and $1225x^2$ denote the numbers.

Then $x^2 + \frac{1}{5}x$, $x^2 + \frac{1}{7}x$, $x^2 + \frac{1}{35}x$ must be squares.

Whence $p = \frac{19}{140}, x = \frac{19^2}{66 \times 140} = \frac{361}{9240}$,

and the numbers are $\left(\frac{361}{1848}\right)^2, \left(\frac{361}{1320}\right)^2, \left(\frac{361}{264}\right)^2$.

Ex. 4.—To find three squares, whose sum may be a square, and if their sum be increased by each of the roots, the sums may be squares.

App.—Let $4x^2$, $9x^2$, and $36x^2$ denote the numbers.

Then $x^2 + \frac{2}{49}x = \square$, $x^2 + \frac{3}{49}x = \square$, $x^2 + \frac{6}{49}x = \square$;

whence, $f = -\frac{23}{1372}, x = \frac{f^2}{2f + m} = \frac{529}{13720}$;

hence the numbers are, $\left(\frac{1058}{13720}\right)^2, \left(\frac{1587}{13720}\right)^2, \left(\frac{3174}{13720}\right)^2$.

If we put $x = \frac{1}{y}$, the expressions become, $my + 1$, $ny + 1$, $py + 1$.

Where the value of f is the same as before, and $y = \frac{1}{x} = \frac{2f + m}{f^2}$

this can also be applied to many problems.

Ex. 5.—To find four numbers such, that the product of any three increased by unity, may be squares.

App.—Here $abc + 1 = \square$, $abd + 1 = \square$, $acd + 1 = \square$, $pcd + 1 = \square$.

Put $bc = m$, $bd = n$, and $cd = p$.

Then $am + 1 = \square$, $an + 1 = \square$, $ap + 1 = \square$,

whence $f = \frac{mn - \frac{1}{4}(m + n - p)^2}{m - (n + p)}, a = \frac{2f + m}{f^2}$.

Now, if $b = \frac{1}{2}$, $c = 2$, $d = 3$. Then $pcd + 1$ is a square.

Hence, $m = 1$, $n = \frac{3}{2}$, $p = 6$,

$$f = \frac{25}{104}, a = \frac{2f + m}{f^2} = \frac{154}{104} \times \frac{104^2}{25^2} = \frac{16016}{625}.$$

Hence the four numbers are $\frac{1}{2}$, 2, 3, and $\frac{16016}{625}$.

The Editor has also to acknowledge the receipt of several other interesting and valuable papers from Mr. Floy, upon the Theory of Numbers, which the limits of the present number will not allow him to insert.

NO. V.

Solution* of an Unsolved Problem,

Proposed in a Scientific Journal, of 1818.

By Professor THEODORE STRONG, Rutger's College, New-Brunswick.

PROBLEM.

It is required to investigate the general equations of the surfaces, at any point of which the sum of its inclinations to the horizon is a constant quantity; one of these inclinations being taken in a plane parallel to the meridian, and the other in a plane parallel to the prime vertical; and to point out the two simplest species of surface having the specified condition.

SOLUTION.

Let the common sections of the meridian and horizon, the prime vertical and horizon, the prime vertical and meridian, be taken for the axes of x, y, z, respectively.

Let
$$z = Ax + By + c, \tag{1}$$
denote the equation of a plane which touches the required surface. By (1) $c = z - Ax - By$ $\therefore$ c depends on z, x, y, A, B, or since at the point of contact z, x, y, belong to the surface touched, they are connected by the equation of that surface, $\therefore$ at that point z is a function of x and y; hence, c depends on x, y, A, B : let this dependence be denoted by $c = F(x, y, A, B)$ = a function of x, y, A, B, then (1) is changed to
$$z = Ax + By + F(x_{,}, y_{,}, A, B); \tag{2}$$
$x_{,}$, $y_{,}$ belonging to the point of contact with the required surface. By taking the partial differentials of (2), first with respect to x, then to y, there result the equations
$$\frac{dz}{dx} = A + \frac{dF(x_{,}, y_{,}, A, B)}{dx}, \quad \frac{dz}{dy} = B + \frac{dF(x_{,}, y_{,}, A, B)}{dy}$$
now, $\frac{dz}{dx}$ = the tangent of the inclination of the touching plane to the horizon, when estimated in a plane parallel to xz; similarly $\frac{dz}{dy}$ = the tangent of the plane's inclination to the horizon when estimated in a plane parallel to yz. Now, these inclinations are evidently the same at every point of the touching plane, and $\therefore$ z, x, y

* This is the solution sent by Professor Strong, in 1818, to Kirk & Mercein, the publishers of the Journal; but from some unknown cause, *no solution was ever published.*

in the above equations may be changed into $z_{,}$, $x_{,}$, $y_{,}$, the co-ordinates of the point of contact with the required surface; but as is well known

$$\frac{dz}{dx} = \frac{dz_{,}}{dx_{,}} = \text{A}, \frac{dz}{dy} = \frac{dz_{,}}{dy_{,}} = \text{B};$$

hence the above equations become

$$\frac{d\text{F}(x_{,}, y_{,}, \text{A}, \text{B})}{dx_{,}} = 0, \quad \frac{d\text{F}(x_{,}, y_{,}, \text{A}, \text{B})}{dy_{,}} = 0;$$

these equations will give $x_{,}$, $y_{,}$ in terms of A, B; hence $\text{F}(x_{,}, y_{,}, \text{A}, \text{B})$ = a function of A and B. Let $f(\text{A}, \text{B})$ denote this function, then (2) becomes

$$z = \text{A}x + \text{B}y + f(\text{A}, \text{B}). \tag{3}$$

By changing z, x, y into $z_{,}$, $x_{,}$, $y_{,}$, and A, B, into $\frac{dz_{,}}{dx_{,}}$, $\frac{dz_{,}}{dy_{,}}$, there results the equation

$$z_{,} = x_{,}\frac{dz_{,}}{dx_{,}} + y_{,}\frac{dz_{,}}{dy_{,}} + f\left(\frac{dz_{,}}{dx_{,}}, \frac{dz_{,}}{dy_{,}}\right) \tag{4}$$

which is the partial differential equation of the surface touched by the plane represented by (3), the co-ordinates of the point of contact being $z_{,}$, $x_{,}$, $y_{,}$.

Now, to apply what has been done to the proposed question: let T = the tangent of the given sum of the inclinations; then, since A = the tangent of the inclination in the plane parallel to $x\,z$, there results $\text{B} = \frac{\text{T} - \text{A}}{1 + \text{AT}}$ = the tangent of the inclination in a plane parallel to y, z; by substituting this value of B in (3) there results the equation of the tangent plane, then by changing x, y, z, A, into $x_{,}$, $y_{,}$, $z_{,}$, $\frac{dz_{,}}{dx_{,}}$ (as in obtaining (4) from (3),) there will result the partial differential equation of the surface touched, and it is evident that the tangent plane and the surface touched are the surfaces required in the question, since they have a common element at the point of contact. If A is supposed to be invariable, we shall have parallel planes for one of the species of surfaces required. To obtain the other species, let all the planes (which are such that the tangent of the sum of their inclinations = T,) be supposed to pass through a given point whose co-ordinates are x', y', z', then will the general equations of these planes be represented by

$$z' - z = \text{A}(x' - x) + \left(\frac{\text{T} - \text{A}}{1 + \text{AT}}\right)(y' - y), \tag{5}$$

which will answer for the second species required; or perhaps the proposer intended the surface formed by the successive intersections of all the planes represented by (5). To obtain the surface mentioned, let A in (5) be changed to A + dA, (x, y, z being constant), and there results (after reduction)

$$1+AT=\frac{\sqrt{(y'-y)\times(1+T^2)}}{x'-x}, \text{ or } A=\frac{\sqrt{(y'-y)\times(1+T^2)}-\sqrt{x'-x}}{T\sqrt{x'-x}}$$

by substituting this value of A in (5), there results the equation

$$z'-z=\frac{2\sqrt{(x'-x)\cdot(y'-y)\cdot(1+T^2)}-(x'-x)-(y'-y)}{T}, \quad (6)$$

for the species of surface sought.

NO. VI.

Solution to the Prize Question,

Proposed in Nash's Ladies' and Gentlemen's Diary, or United States Almanac, No. III, *for* 1822.*

BY NIMROD COLBURN, Esq. American Citizen.

QUESTION XXI. (51.)—or Prize Question.—*By W. Marratt, A. M. Liverpool, England.*

Let a cone, the altitude of which is 6, and base diameter 2 feet, be filled with water, the sides of the cone being indefinitely thin, and of the same specific gravity as water; let this cone be suspended by its vertex, and made to vibrate like a pendulum; if an orifice of one-tenth of an inch in diameter be opened in its base, in what time will the vessel vibrate, after the water has been flowing for *one* minute?

SOLUTION.

Since the vibrations are of small extent, and the velocity of the cone very small, the effect of the centrifugal force upon the flowing of the water will also be so small that we may neglect it entirely, and consider the cone as at rest, the water running out from a circular hole of one-tenth of an inch in diameter at its base, for one minute.

Put $g=32.2$ feet, $a=$ the area of the orifice, S $=$ the solidity of the whole cone, Q $=$ the quantity of water or fluid cone discharged in any variable time (t) from the commencement of the motion, $h=$ the height of the whole cone, $x=$ the height of Q.

Now, (by known principles of hydrostatics) the velocity with which the water runs out at the orifice, is the same as that which a heavy

* We are not aware that solutions have been published to any of the new questions in Nash's Almanac, No. III, since the work was discontinued after that year.—ED.

body would acquire by falling half the distance from the surface of the fluid to the base ; let V denote the velocity, then

$$V=\sqrt{g(h-x)}.$$

Since $aVdt=d\,Q=$ the quantity of water discharged in the time dt, I have

$$dQ=a\ \sqrt{g(h-x)}\times dt; \tag{1}$$

but similar cones are as the cubes of their heights,

$$\therefore x=h\left(\frac{Q}{S}\right)^{\frac{1}{3}},$$

hence (1) becomes,

$$dQ=a\ \sqrt{gh\left(1-\left(\frac{Q}{S}\right)^{\frac{1}{3}}\right)}\times dt,$$

let

$$a\sqrt{gh}=A,$$

then

$$Adt=\frac{dQ}{\sqrt{1-\left(\frac{Q}{S}\right)^{\frac{1}{3}}}} \tag{2}$$

In order to integrate (2), put $\frac{Q}{S}=y^3$, and it gives

$$\frac{A}{3S}\,dt=\frac{y^2dy}{\sqrt{1-y}},$$

whose integral, after being corrected, is

$$\frac{A}{3S}\times t=\frac{16}{15}-\frac{2}{5}(1-y)^{\frac{5}{2}}+\frac{4}{3}(1-y)^{\frac{3}{2}}-2(1-y)^{\frac{1}{2}}; \tag{3}$$

by substituting in this last equation for A and S their values, and by putting $t=60$, y can be found by the usual methods of trial and error, and thence Q becomes known : or by substituting in (3), for y its equal $\frac{x}{h}$, x can be found, and thence Q. But it is evident from the question, that Q is very small relatively to S: if, therefore, in (2) $\left(\frac{Q}{S}\right)^{\frac{1}{3}}$ is neglected relatively to (1), then, by integration,

$$At=Q \text{ nearly};$$

whence Q will be found very nearly ; which, however, may be accomplished with still greater exactness, by converting (2) into a series.

Having found Q as above directed, find its momentum of inertia and that of the whole cone about the axis of suspension, then divide the difference of these momenta by the product of the difference of the cones $\times$ the distance of its centre of gravity from the point of suspension, and the quotient will be the length of a pendulum which will vibrate in the same time with the frustum, whence the time of vibration becomes known. The remainder of the process having no difficulty but its length, I shall omit it, since it would but unprofitably occupy the pages of this journal.

ARTICLE XXXIII.

JOSEPH LOUIS LAGRANGE.

The instances in which extraordinary intellectual powers, evinced very early in life, have been retained to a ripe old age, are few in number and sparsely recorded. The efforts of youthful genius seem, like the unseasonable germinations of the plants of the earth, to fall to the ground, while they are yet blown, without bringing aught to maturity; and we are thus often called upon to exclaim in mournful admiration of the relics,—these, then,

"Are all that must remain of thee."

A remarkable example, however, of early genius and of continued mental activity, is to be found in the life of the distinguished Piedmontese whose name heads our page. This eminent mathematician was born at Turin, early in the year 1736, of a family of French extraction, and one of some consequence and wealth. The imprudence of his father had dissipated his fortune, and was, perhaps, the cause which determined the son to the study of mathematics as a profession.

Lagrange manifested early in life, an inclination for classical studies, from which he was accidentally diverted by the perusal of a paper, written by Dr. Barrow, on the great powers of the new Geometry. This predisposition is not altogether anomalous and isolated; for a similar turn of mind was observed in his celebrated friend and successor in analytical science, the author of the *Mécanique Céleste;* and we must ever esteem it an highly fortunate circumstance, that these two eminent individuals were induced to leave those pursuits, in which their services in the cause of human happiness would have been, perhaps, in a great measure lost; and to direct their strong and original minds to the advancement of mathematical science, and to its applications to the explanation of physical phenomena.

While he was yet but eighteen years of age, Lagrange addressed a letter to M. Fagnano, in which he lays down formulas which serve to express the differentials and integrals of any order, by means of a general series analogous to Newton's Binominal Theorem. About this period, he was also placed at the head of the mathematical department of the Military School, the duties of which station, notwithstanding his youth, he discharged with credit to himself and satisfaction to his patrons.

It was by his exertions, in conjunction with those of a few friends, that the Academy of Turin was established in 1757; to the memoirs and publications of which, from 1759 to 1785, embracing a period of nearly thirty years, he was a constant contributor. The mathematical tracts alone which he has there given to the world, would have been sufficient to have insured for him a high name among the most successful cultivators of the science. They are all written by a master

hand, and comprise many very interesting discussions. Among them, we should name those, *on the nature and propagation of sounds*, in which he puts beyond all doubt the generality of the solution of D'Alembert, in regard to the vibrations of chords, as had also been previously shown by Euler; *on the maxima and minima of indefinite integrals*, a paper which immediately ranked its author among the first mathematicians of his day, and which contains the genuine and peculiar algorithm of his Calculus of Variations, as it was termed by Euler. The most mixed questions are here taken up and solved; the brachystochronous curve, the principle of the living forces of mechanical systems, and the law of least action, which last he generalizes, and shows to be sufficient to determine all the problems of matter in motion.

In 1764, he gained the prize offered by the Academy of Sciences at Paris, for the best dissertation on the Physical Causes of the Librations of the Moon; and two years subsequently he was, through the influence of Euler and D'Alembert, invited by Frederick the Great to assume the direction of the Academy at Berlin, from which the former had just retired. His early publications had attracted the attention and won for him the friendship of that distinguished philosopher, with whom he opened a correspondence as early as the year 1754, and by whose exertions he was elected a foreign member of the Berlin Academy. From this time until 1763, Lagrange was a contributor to the Berlin Memoirs, and at the same time continued his publications in the transactions of the Turin and Paris Academies. The limits, however, which we have prescribed to ourselves in this sketch, preclude any further notice of them.

His *Mécanique Analytique*, the great work on which his fame will rest, was first published in 1788, although he had had the project in view for nearly twenty years previous. This, like many other publications destined for immortality, was refused by the booksellers; and was at length published solely upon the condition that his friend the Abbé Marie should undertake to bear a moiety of the loss, in the event of failure. It is, as is well known, based on the principle of virtual velocities, and of a purely analytical character; and as he himself informs us in his preface, does not contain a single diagram.

On the demise of his patron, King Frederick, he removed to Paris, where his reputation had long preceded him; and continued to reside there during the troublous times of the revolution. Wisely abstaining from the political conflicts of the day, he insured his own safety and gained the respect of all parties. The subject of weights and measures which was now agitated, and which had interested all the scientific men of the French kingdom, found in Lagrange a warm, determined, and uncompromising advocate of the system of decimation.

The violence of the revolutionists had at one time nearly reduced him to leave France; but the establishment of the Normal School, in which he was appointed a professor, had the effect of changing his destination. It was at this period, 1797, and while engaged in his old occupation, that he gave to the world his Theory of Analytical Func-

tions, (*Theorie des Fonctions Analytiques*;) and in the ensuing year, his Resolution of Numerical Equations, (*Resolution des Equations Numeriques*) as well as his Lessons on the Calculus of Functions, (*Leçons sur le Calcul des Fonctions*) in which he contends that the higher analysis can be fully explained on *algebraical* principles.

Nor did he confine his attention to the mathematical sciences exclusively; he was hardly less a proficient in chemistry and botany, to which, and to the fine arts, he at one time gave almost his sole attention; sensible, perhaps, of the undue unequality of mind which a constant and unvaried devotion to one particular pursuit, and that, so exact a science as mathematics, never fails to produce. Neither is it to us surprising, that one who was so awake to the harmony of numbers, should be equally alive to that of sweet sounds; it is said of him that so congenial were the two to his mind, that he was always lulled into a train of mathematical reflection, after hearing the first strain of a piece of music, and that its continuation seemed as but an assisting accompaniment to his mental calculations.

The evenness of temper for which he was so remarkable, gained him many warm as well as admiring friends. Ever kind to the unfortunate, as he was attached to his intimates: bland and courteous in social life as he was calm and respectful in controversy, he seems to have gone on in the even tenor of his way, beloved for his virtues and honoured for his talents. The list of titles which we find affixed to his name, show how highly he was esteemed by the land of his adoption, the birth-place of his ancestors: *Mèmbre du Sénat Conservateur, Grand Officier de la Legion d'Honneur, et Comte de l'Empire*, are some of the dignities conferred on him. The last seven pages of the *Méc. Anal.* vol. ii. are occupied with a list, communicated by M. Lácroix, of his printed works, papers, &c.; upwards of an hundred memoirs on various subjects in abstract and physical science are there enumerated.

His death, which took place on the 10th of April, 1813, was hastened by his great exertion in preparing a new edition of his Mécanique Analytique; the fatigue was too much for a frame worn out by scientific labours and a constitution naturally delicate. He was buried at the church *de St. Genevieve*; and the last tribute of honour and respect to his memory was paid by Laplace, in a funeral eulogy,—than whom no one certainly was more capable of appreciating the talents or of portraying the virtues of a kindred spirit. Σ.

ARTICLE XXXIV.

Horae Decerptae.

FROM A MATHEMATICIAN'S DIARY.

"From grave to gay, from lively to severe."

SCENE.—*The Library—Mathematical Volumes abound—Busts and Portraits of distinguished Analysts ornament—and the* DOCTOR, COLBURN, *and* SIGMA *seated at the corners of an Isosceles Triangular Porcelain Tablet dignify the spacious apartment—a cup of coffee before each.*

COLBURN, (*loquitur.*)

One might as well attempt the equation of the track of a woodchuck in a cane-brake;—let me see,—by eliminating t, I have x equal to the square root of the square of *am* into the cube——

DOCTOR.

All wrong, wrong, wrong! start again.

COLBURN.

Not at present, doctor; my brain is just now an unknown quantity with a co-efficient equal to nought——

SIGMA.

—— divided by nought equal to one.

DOCTOR.

Patience, or, as Sir Isaac says, severe and patient industry is the darling virtue of mathematicians; it is nonsense to expect to do any thing without it.

COLBURN.

You will allow, though, that such are the imperfections of our nature, that even philosophers may be tempted from the rigid path of this virtue.

SIGMA.

Hist! the doctor's in the imparting mood.

DOCTOR.

Now, young men, (for you are not of threescore and upwards and of numbered days) let me recommend as a motto by which you may quadrate your studies, the old one—*festina lente.*

COLBURN.

Admirable!

DOCTOR.

Never read a mathematical work without understanding every portion of it thoroughly before you throw it aside; if difficulties start up, mark them,(1) and return to them when you shall have mastered the more easy parts. But never raise the siege after you have once set yourself regularly down to it; spend days, nights, and weeks, and even months, if necessary. You will be the gainer, and in the end save time.

COLBURN.

That love of paradox.

DOCTOR.

I repeat, it matters not if you give *time* to a subject, provided you DO understand it. The difficulty once overcome, you are advanced one step farther towards your *couche de niveau.*

COLBURN.

This, doctor, is all pretty in the abstract; but in practice, you know, there are always some peculiar circumstances which falsify our theoretic results. Now, flesh and blood is not always a constant quantity,—nor is the mind. There are changes of temperature, resistances of solids and fluids, external forces of action, pressures, accidental collisions, &c. &c. in these operations of the mind, and body, too, as well as in those of nature.

DOCTOR.

A young mathematician should not grasp at too much at first; it is absurd to expect to acquire all that has been written and done upon any great subject, without more years than are allotted to man. Life will be frittered away to no purpose, if the novice haste too much. Ambition is a passion that as often vauntingly o'erleaps itself in letters as in politics.

SIGMA.

Precepts worthy of inscription on a golden tablet!

DOCTOR.

Read! read!! read!!! and let your reading be of some elementary work of established character.

COLBURN.

What think you, doctor, of Delambre's Large Astronomy?(2)

DOCTOR.

A finished model for all mathematical writers of this kind. You

(1) Thus (?).
(2) Traite Complet D'Astronomie, 3 vols. 4to. Paris, 1814.

are not continually forced to verify his results by actual calculation of your own ; but each step is there distinctly laid down and taken.

COLBURN.

I would run a parallel between Delambre and Dr. Bowditch in this respect ; both are ever solicitous to give full investigations.

DOCTOR.

There is also *Arbogast's Calculus of Derivations*, a most excellent elementary treatise, whose only fault is, perhaps, too great diffuseness after the first fifty or sixty pages ; also Monge, dear, delightful Monge ! I should compare his Analytical Geometry(3) to an overflowing cup of this exquisite coffee.

COLBURN.

You recommended me, Sir, when I first knew you, to read Gauss' *Theory of the Celestial Motions*,(4) and I can't remember any volume whose perusal gave me more pleasure.

SIGMA.

And I, doctor, by your advice attempted his *Theory of Numbers*.(5) I got about half through, and then threw it aside for Dr. Bowditch's first volume.

DOCTOR.

Both most excellent and profound works : the first is remarkable for its simplicity, and the second for the elegance with which he has treated a subject so abstruse, and to many dull ; but the learned professor has lately created a new treatise upon the *Shape of the Equilibrium of Fluids*,(6) which also bears the strong impress of his powerful and original mind. He is a most extraordinary man, and was justly called by Lagrange "Le premier des mathematiciens."

SIGMA.

What on the Integral Calculus ?

DOCTOR.

Euler's is undoubtedly the best introductory work on the sub-

(3) "Application de L'Analyse a la Geometrie," 4to. Paris, 1807. This, together with M. Dupin's excellent supplement to it, entitled "Developpemens de Geometrie," 4to. Paris, 1813, form a most complete treatise upon the Geometry of Curved Surfaces. The last edition of Monge is out of print, and can seldom or never be procured ; Bachelier, the enterprising successor of Madam (the widow) Courcier, has a new edition in press ; may it soon appear.

(4) "Theoria Motus Corporum Cœlestium, in sectionibus conicis Ambientium Solem," 4to. Hamburgh, 1809.

(5) Disquisitiones Arithmeticæ," 8vo. Leipsic, 1801. "Recherches Arithmetiques," translated from the former by M. Poullet Delisle, 4to. Paris, 1807.

(6) "Principia Generalia Theoriæ Figuræ Fluidorum in Statu Æquilibrii," 4to. Gottengen, 1830.

ject(7)—an old treatise, but an undying one. But, gentlemen, I see my servant shuffling this way; so let us have a moment's exercise on the tablet before he arrives. If neither of you have examined the last leaf of Legendre's *Exercises in the Integral Calculus*,(8) you will be obliged to me for introducing you to a curious proposition which he has there succinctly demonstrated.

COLBURN.

Gratias tibi!

DOCTOR.

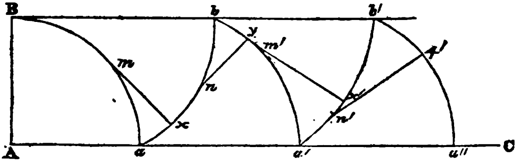
Required the resulting evolute of an infinite series of evolutes.

SIGMA.

How produced?

DOCTOR.

Draw two rectangular axes AB, AC, thus;—(*draws the figure*)—



let B*ma* be any *rectangular* curve, *id est* such that tangents at its extremities B and *a* will one be parallel and the other perpendicular to AC; now suppose B*mx*, a thread which, upon being unwrapped, will describe the evolute of B*ma*, and——

COLBURN, (*aside to Sigma.*)

It takes an awful deal of hypothesis to get over the doctor's diagram.

DOCTOR.

——that *bm'a'* is the evolute of this last counted from *b*, and so on *ad infinitum*; now the last term of the successive *developements* of B*ma*, will be the curve which is asked.(9)

(7) "Institutiones Calculi Integralis," (best edition) 4 vols. 4to. St. Petersburgh, 1792—94, price $20 to $24. The fourth volume, however, is rarely to be found, the only copy we know of in this country being in the possession of Dr. Bowditch, of Boston.

(8) "Exercises de Calcul Integral," "Construction des Tables Elliptiques," 3 vols. 4to. Paris, 1811—19, with supplements, (referred to at page 254 of the present number.)

(9) The required curve is a cycloid, as will be seen from the subjoined analysis; for we cannot refrain from giving this truly remarkable theorem a passing notice To John Bernoulli is due the merit of its first discovery, (see the fourth volume of his works, *Lausannæ et Genevæ*, 1742, &c.); but the first demonstration of it was given by Euler, in the Memoirs of the Academy of St. Petersburgh, for 1764; also it is remarked by M. Poisson, (*Journal de l'Ecole Polytech-*

COLBURN.

An odd kind of a problem, in good sooth; and——(*Enter* DIOCLES, *with a bundle of books and papers, which he deposits ante* SIGMA, *and then wheels off to his station under the bust of his illustrious prototype of cissoidal memory.*)

COLBURN.

Here they are at last! the *Bulletin des Sciences*, the *Annales de Mathematiques*, the *Mémoires* et hoc omne Genus.

nique, cah. 18. p. 440.) that an elaborate memoir upon the subject may be found among the *manuscripts* of Lagrange, deposited in the Library of the Institute, which appears to have been read before the Berlin Academy, but which from some unknown cause has never been printed.

M. Poisson has presented an extremely direct demonstration of this property of the cycloid, in the Journal referred to above, among many problems proposed and resolved for the purpose of showing the use of the transformation of functions into a series of periodical quantities.

Let R denote the radius of curvature, and x, y the co-ordinates of any point in the required evolute, m the increasing degree of the successive developements, θ the angle formed by a tangent at that point with the upper parallel, and a the perpendicular distance between the two.

As m increases, the different terms of the series contained in three formulas, given by M. Poisson, for the values of R, x, y, (which our limits will not allow us to introduce) diminish indefinitely; they approach constantly nearer to zero; and if m be supposed infinite, they entirely vanish; so that at this limit they are reduced to

$$R = \frac{4a}{\pi}\sin.\theta, \; x = \frac{a}{\pi}(1 - \cos.2\theta), \; y = \frac{a}{\pi}(2\theta - \sin.2\theta). \quad (c)$$

The curves which are deduced from each other by successive developements, and which are always contained between two parallels, tend continually towards a constant form, which can only be attained after an infinite number of developements, but from which they differ less as this number becomes greater. This limiting curve, which corresponds to equation (c), may be easily recognised to be a semi-cycloid, whose base is perpendicular to the two parallels, and equal for the whole cycloid to $2a$.

In fact, if θ be eliminated between the values of x, y, and if after differentiation, we make $\frac{a}{\pi} = b$, we find,

$$dy = \frac{xdx}{\sqrt{(2bx - x^2)}}$$

which is the differential equation of a cycloid having its base upon the axis of y, and the radius of its generating circle $= b$; which supposes its base $= 2\pi b$, or $2a$. Σ.

SIGMA.

Not so fast, my dear Colburn. These are not the French periodicals.

DOCTOR.

No, it is said that every evil brings its good; and I am inclined to think the converse no less true, that every good has its evil. The *trois jours*, so fraught with beneficial results in other things, have caused science and letters to stagnate; this it is, I opine, that has caused the delay in receiving the works you have enumerated.

SIGMA.

We can hardly reconcile the disposition evinced in the glorious revolution, or more properly revolutions, with the calmness and placidity of scientific genius.

COLBURN.

Look at the example of Archimedes; he was found in his study even after the gates of the city were forced by the enemy,—yet it was just before that he himself had repulsed a hostile fleet.

DOCTOR.

The alacrity in action evinced by the Franks, is what would be expected from their rapid perception of the abstruse and dark. There is always a certain irritability in your true genius, which causes him to be no less prompt to act than he is quick to see.

SIGMA, (*aside to Colburn.*)

The doctor is too fond of these metaphysics.

COLBURN, (*to Sigma.*)

Or, as Goldsmith said of Burke,

"Too deep for his hearers, he goes on refining,
"And thinks of convincing, when we think of dining."

DOCTOR.

To sum up all in a word,—mathematical genius is the highest grade of mind. It combines the excellencies of many others, and yet has a peculiar and saving property of its own.

SIGMA.

Would you say, then, that great mathematicians are born with a predisposition, or greater capacity of mind for these pursuits?

DOCTOR.

I adopt a middle opinion in regard to this subject—*in medio tut*, &c.;—that we are born with innate tastes, but that industry and application only will develop them fully, that assiduity will do much to generate a taste for other pursuits, and that even without what is vulgarly called genius, we may accomplish much, nay, all, by *perseverance* and ATTENTION. The variety of minds results, therefore——

SIGMA.

Mr. Colburn is a journeying to the land of Nod.

COLBURN, (*starting from his chair, rubbing his eyes, and giving a blissful yawn.*)

No! no! oo, oo, wide awake, I assure you. You were saying that the evolute would be a queer kind of a curve, and——

DOCTOR.

Not I, sir; I was observing that the difference between minds was as much the result of habit as of inborn capacities.

SIGMA.

Guilty of contempt of this tribunal.

COLBURN.

I crave your mercy.

DOCTOR.

Which I extend to you, on condition that you hand me that package of books. (*Colburn gives him the books laid on the table by Diocles.*) What are these?

SIGMA.

The latest mathematical works.

DOCTOR.

Let us read over the list:—*Imprimis*, Mr. Grund's Translation of Hirsch's Algebra;(10) decidedly Bostonian in its appearance and typography. Translations of M. Bourdon's Algebra, by Lieutenant Ross, of West Point,(11) by Professor Farrar, of Harvard,(12) and by Professor De Morgan, of the London University;(13) the last containing but the three first chapters of the original. Then we have Dr. Jamieson's work,(14) introductory to Bland's Algebraical Problems; the ninth edition of the Treatise of Wood;(15) and lastly, the Self-examinations in Algebra of Mr. Wright.(16) Quite enough, I should think.

COLBURN.

But all valuable works.

SIGMA.

I should like to pay the last a compliment for his high abilities and great acquirements.

COLBURN.

He has amply dignified his AB-*ship*, by his *Solutions to the Cambridge Problems*,(17) as well as his *Commentary* on the *Principia* of Newton.*

(10) 8vo. Boston, Mass. 1831.
(11) 8vo. New-York, 1831.
(12) 8vo. Cambridge, Mass. 1831.
(13) 8vo. London, 1828.
(14) 8vo. London, 1830.
(15) 8vo. Cambridge, Engd. 1830.
(16) 8vo. London, 1825.
(17) 2 vols. 8vo. London, 1825.

* London, 2 vols. royal 8vo. 1828.

DOCTOR.

Mr. Wright is truly of great promise.

SIGMA.

Here also are *Mr. Airy's Mathematical Tracts.*(18)

DOCTOR.

That on Physical Astronomy, written, too, before he had attained his eighteenth year. I would class him with Clairaut and Pascal; though the early promise has not as yet rewarded us with rich fruit. The tract on the Calculus of Variations is excellent.

SIGMA.

England, indeed, seems to be doing nothing for the pure mathematics. But Russia is said to be making rapid advances towards high analytical superiority. The papers of M. Ostrogradsky, a second Euler, fill the Petersburgh transactions.

DOCTOR.

Mr. Ivory, who perhaps ranks highest in the land of reform, has taken up the *Theory of Elliptical Transcendants*;(19) a difficult subject, which he will no doubt greatly simplify, and this is about all that we are to expect from her. How has the sea-girt island fallen!

COLBURN.

"Fallen from her high estate."

SIGMA.

Have you seen the work of M. Jacobi, doctor?

DOCTOR.

On Elliptic Functions?(20)—yes—and purpose giving an article on it in an early number of * * * * * * * *'s Journal.

COLBURN.

Here is also *M. Lagarrigue's Trisection Compass.*(21)

DOCTOR.

A pamphlet of ingenuity, which, however, I regret is not spent on a more useful subject.

SIGMA.

But what does the genius of Columbia? Is she not sufficiently taught in the mathematical sciences to win laurels for herself?

DOCTOR.

That she is; and let me mention first and foremost, the American Mécanique Céleste, volume second.

(18) Mathematical Tracts on Physical Astronomy, the Figure of the Earth, Precession and Nutation, and the Calculus of Variations, 8vo. Cambridge, Engd. 1826.

(19) Philosophical Transactions of the Royal Society of London, Part II, 1831, Art. V.

(20) Fundamenta Novæ Theoriæ Functionum Ellipticanum, large 4to. 1829.

(21) The Compass of Proportions Improved; or, in other words, Construction of a Mathematical Instrument, whereby an arc may be divided into three equal parts, 8vo. New-York, 1831.

SIG. ET COLB.

The American Mécanique Céleste!

DOCTOR.

It is a brilliant undertaking: magnificent in the design, and no less so in the execution; splendid, too, in its typography;([22]) and in every respect a patriotic and worthy exertion in the cause of American science; and this second volume of Dr. Bowditch is no less admirable than the first.

COLBURN.

There has been a fluttering among the stooled pigeons of Europe at this achievement of our countryman; and the automata professors there, begin to open their eyes at last to see what has been snatched from them.

"Alas! poor Yorick!"

DOCTOR.

The Commentary on the *Mécanique Céleste*, is the elaboration of no ordinary mind; but of one deeply taught in the mysteries of mathematical philosophy. It will be studied, read, and remembered always in connexion with the great work which it embalms; it is also a distinct treatise on all the improved methods of the analysis of our day, of which none have made a better use than Laplace. How great is our debt of gratitude to the American commentator!

SIGMA.

Just allow me, doctor, to ask you three questions about this same second volume.

DOCTOR.

Proceed, sir, and I will reply to the best of my ability.

SIGMA.

First, as to the partial differential equation for the attraction of an ellipsoid, contained in volume second, page 15, where the text of Laplace runs thus, "Il est aisé de s'assurer par la differentiation qu'on aura entre les quatre quantités B, C, F et V l'equation suivante"——

COLBURN, (*interrupting him.*)

I should like to know what Laplace means there by *aisé*.

DOCTOR.

I am surprised that you should be ignorant of the fact, that throughout the whole of the *Méc. Céleste*, wherever that word occurs, you may brush up for a week or two of no ordinary labour. Why, sir, with me it is proverbial.

(22) We have had the pleasure of seeing a letter from M. Lacroix, to a distinguished scientific American, in which he mentions the Commentary and Translation of Dr. B. as the most beautiful specimen of mathematical typography now extant. "Le plus beau livre de mathematiques qui a jamais ete imprime."

SIGMA.

——"à differences partielles," &c. How has Dr. B. enlightened that difficulty, *à-priori?*

DOCTOR.

No, indeed; the à-priori demonstration would occupy fifty pages instead of the two into which he has condensed his own singularly beautiful and lucid one. (*The* DUO *exchange simultaneous glances of wonder and astonishment.*)

SIGMA.

Next, as to the Ivory and Poisson question of the figure of the Planets; in whose favour has he decided? for upon that depends the fate of our countryman.

DOCTOR.

You say truly; but how can you doubt as respects his decision, which is most assuredly upon the disputed point in favour of M. Poisson; to a complete developement of all of whose valuable papers upon the subject, *he* has added an original demonstration, which, like himself, is neat, simple, and convincing.

SIGMA.

Lastly, as to the phenomena of the Tides, Precession, and Nutation; what says the doctor of these most difficult subjects?

DOCTOR.

Don't mention them, unless you would drive me into a rhapsody, which——

SIGMA.

Heaven forefend!

DOCTOR.

The two *Mécaniques* have left little to be desired in the geometry of matter; that little will soon be supplied by the new Philosophy of Poisson, thus——

SIGMA.

A new philosophy!

DOCTOR.

Completing the whole theory of Philosophical Mechanics, the most important subject in nature which can occupy the attention of man.

SIGMA.

A pinch of Lundifoot, if you please, Mr. Colburn;—kish!—kish!—kesho!—ke-eshoo!

COLBURN.

May you not live a thousand.

SIGMA.

What is this new philosophy of which you speak, my dear doctor?

DOCTOR.

Right sorry am I, that I cannot explain it fully; truths that dawn upon great minds are only to be clearly developed by them. Lagrange has given us the theory of mechanics, as the term is vulgarly understood, founding his system upon the principle of virtual velocities, which, as you are well aware, was first given to the world in a letter written by John Bernoulli, and published by Varignon, in his *Projet de la Nouvelle Mécanique.*

COLBURN, (*aside to Sigma.*)

The doctor is pedantic.

SIGMA, (*aside to Colburn*)

'Tis his old foible.

DOCTOR.

Laplace took a bound beyond him, and transcribed on a few leaves the history of the universe for ages to come; his eagle-eye stretched over all space, through all time; he robbed the heavens of their mysteries; demonstrated how vain, though sublime, their terrors; in a word, made us familiar with all the magnificent operations in the universe, past, present, and to come, bringing all down to a NOW.

COLBURN.

Requiescat in pace!

DOCTOR.

But there is left an equally wide field for exploration on the other side of the *Mécanique Analytique.* The minute as well as the grand has its interest. Nature is to be considered, examined, and conquered in her microscopic forms; the doctrine of the Leibnitzian monads is yet to be examined. And who living is more fitted for such a task than M. Poisson?

COLBURN.

"Hail to the chief."

DOCTOR.

Who more capable of giving us the analysis of this ATOMIC PHILOSOPHY?

* * * * * *

SIGMA.

What else are we to expect?—What of the Col. Coll. Mechanics?

DOCTOR.

The work of Professor Renwick,(23) you mean. Any thing from

(23) Elements of Mechanics, by James Renwick, LL.D. Professor of Natural Experimental Philosophy and Chemistry in Columbia College, New-York; Philadelphia, Carey & Lea, 1832.

that source is entitled to consideration; it will, doubtless, be *American*, as far at least as a treatise of this kind can be made national. Our elementary works on the pure and physico-mathematical sciences, have savoured too generally of a foreign mint It would certainly be more creditable to us as a nation, and becoming us as an independent people, to rely less upon bald translations and compilations of, and from transatlantic publications, and more upon our own exertions. We ought and must think for ourselves, and if we cannot advance any thing new, we may at least be original in disposing and arranging the materials of science.

COLBURN.

And is Professor Renwick's book of this kind?

DOCTOR.

I have reason to think it will be alike creditable to the country and to himself.

SIGMA.

The Diary, at least, is such a work both in spirit and execution as you would wish. It is meant to be a *porte-feuille* of American genius; and while it invites and would kindly appreciate the contributions of the mathematicians of any clime and under any sky, it claims to be founded on national principles and to have for its object the particular encouragement of mathematical effort among ourselves.

COLBURN.

Long live the Diary!

SIGMA.

No exertions, I am convinced, will be spared to render it a worthy and acceptable depository of the solutions of its correspondents, of whatever celebrity in the world of science. It is free to all—shut to none. *Carpere et Colligere*, may be its motto, but without reference to persons.

DOCTOR.

I can well conceive of its claims to the attention of those who have already honoured its pages with their solutions.

SIGMA.

The Mathematical Diary has been most nobly sustained by the first talent of the land, and will continue to be so sustained! It is a common ground, where the acolyte may tilt and tourney with the belted knight; where spears may be broken with visors down, and where the veteran may joust with the novice, without the fear of compromising his dignity, or of disparaging his known skill.

COLBURN.

The Editor certainly deserves our thanks for his patriotic and unpaid[24] labours and exertions to advance the mathematics among us.

SIGMA.

Ne quid nimis. Doctor, will you favour us with your views upon the poetical and mathematical mind?

DOCTOR.

I am inclined to believe that there is a closer affinity between the poetical and mathematical mind, than is generally supposed; the seeming difference observed, arising from habit. Look at that medallion of Lagrange, and you need not consult J. C. Lavater or Spurzheim, to perceive indications of a superior imaginative faculty—he, it is said, was both poet and musician.

COLBURN.

I must study physiognomy and phrenology, for I can see neither signs nor bumps.

DOCTOR.

The truth is, that the poet and the mathematical philosopher, both give play to the imagination; and differ only in the use which they make of it. The poet skims over the surfaces of things; he looks nature in her eye, and interprets her glances as they affect; he decks his subject with the light, the bright, or the sombre and gloomy, as, glancing "from heaven to earth, from earth to heaven," he can see aught in external nature to serve his purpose; he is pleased with pomp, and show, and circumstance;—with every thing that strikes.

COLBURN.

Well!—the other side of the equation?

DOCTOR.

The mathematician seeks to discover the nature of things, yet has no less power of imagination. The poet looks upon the heavens, and the light above affects him with its changing hue, with the bright spirits which livingly stud its concave. His fancy plays around that which he sees. But our philosopher darts his mind's eye through the empyrean. He does not wonder that such sublimities exist, but is anxious to know how they harmonize. He dallies not with the obvious, but searches into the recondite and abstruse. In fact, the difference is but one of degree; the mathematician discovers realities, and gives them a "local habitation and a name:" the poet stops and gives loose to his fancy upon "airy nothings."

(24) It may be well to observe, that in return for the fifteen hundred dollars which Mr. R. has expended in the publication of the Diary, he has not received above five hundred in subscriptions.

SIGMA.

Yet there seems to be an irreconcileable difference between the two.

DOCTOR.

Habit—all habit; the imagination is only employed on different things.

COLBURN.

You are always obscure, doctor, when you get upon the *metaphysique* of any thing but the *Calcul.*

DOCTOR.

What can be more imaginative than the investigations of Laplace; or what is more calculated to make one " rapt inspired," and to generate poetic feelings, than his flights through the " azure deep of air?" The mind which is alive to the gentlest tremblings of the music chord, and which can stretch its fancy beyond this " diurnal sphere," and conceive of systems of worlds on systems rolled, must have poetry in his soul.

COLBURN.

The poetry of the spheres!

DOCTOR.

A mathematical poet is not an entity of paradox, as many would believe.

COLBURN.

I almost feel the inspiration.

SIGMA.

And I——

DOCTOR.

Adventate Dea. Procul, O, procul este profani
Toto que absistite luco.

[*All ascend and disappear in a rhapsody.*

It was the intention of the writer of the two preceding articles, to have inserted in the present number of the Diary, a copious analysis of all the scientific works and periodicals which have appeared within the past year in Europe and America. But he has been deterred from so doing, by an unwillingness to swell its size, already extended far beyond its accustomed limits.

He, therefore, hopes that the opinions pronounced on various scientific works, in the " Extract from his Diary," will be esteemed by all who examine them as oracular; and farther—that the following notices, drawn blindfold, from his Balaam-box will not be deemed unacceptable.

Σ

ARTICLE XXXV.

Mathematical Investigation

Of the Motion of Solids on Surfaces,

In the two Hypotheses of perfect Sliding and perfect Rolling, with a particular examination of their small Oscillatory Motions. By Henry James Anderson, M. D. Professor of Mathematics and Astronomy in Columbia College, New-York. 4to. pp. 70. Philadelphia, James Kay, Jun. First published in the Transactions of the American Philosophical Society; New Series, Vol. III. Part 2d.

This is a subject of the deepest interest in the higher branches of Dynamical Science, and is treated of in the Memoir now before us, in a manner and to an extent to which it has never been carried by any previous writer. It is well known, that Lagrange, in the second volume of his *Mécanique Analytique*, first gave to the world an investigation of the small oscillatory motions of any system of bodies round the places of their rest; and that after a general investigation of the free rotation of a rigid body, he proceeded to the examination of the well known case in which the body *pirouettes* by virtue of the inertia of the elements alone. After a masterly detail of all the circumstances of this case, Lagrange has also entered upon the discussion of the general motions of a heavy body *pirouetting* about a fixed point, not the centre of gravity, and in the case when the axis of the body is subjected to a series of small oscillations around the vertical. Professor Anderson has, in an extremely able and elegant manner, extended the method of Lagrange to the problem of the motion of a rigid body, bounded and supported by any surface whatsoever, whether it slides or whether it rolls freely upon the fixed surface.

The Memoir commences with a brief but most interesting and complete history of all the researches of mathematicians from the time of Galileo to the present day, upon the motion of solid bodies. He then establishes the differential equations of the problem; by applying his equations to the case in which the two surfaces are of the second degree, he reduces the question to the particular case in which the body is subjected to *small* oscillations alone, employing a property of curved surfaces, demonstrated by Dupin in his supplement to the *Analytical Geometry* of Monge, "That every curve surface may be osculated in each of its points by a paraboloid of the second degree, having the normal for its axis." The right line passing through the centre of gravity of the moving body and the point of contact, will not be in general a permanent axis of rotation; but as the method of Lagrange is wholly independent of the properties of the Segnerian axes, the determination of the motion of the body is not rendered more difficult on that account. Dr. Anderson supposes afterwards that the fixed surface is plane or spherical, that of the body being always arbitrary.

After making the necessary reductions in his formulæ, he from them deduces the differential equations expressive of the small oscillations of the axis of the body and its point of contact; these equations he reduces to four linear equations of the second order, with constant coefficients, which, he observes, may be again reduced to two of the fourth order of eleven terms each, no term being wanting; the latter are integrable by the method of D'Alembert, and give for the variable elements of the position of the moving body and all the phenomena of the motion, expressions composed of the sines and cosines of arcs proportioned to the time. The conditions of oscillatory motion will also be expressed by equations of limitation arising during the process of determining the integrals.

The paper concludes with an application of the preceding formulæ to the determination of the small oscillatory motions of bodies of any figure, law of density and areola of contact, rolling with the three rotations on a surface, which, from slight asperity, or other cause, prevents entirely and in all directions the sliding motion of the body, while in other respects it leaves it free to rock, pitch, and spin, with any combinations of these motions consistent with a small declination of the natural vertical of the body from the vertical of equilibrium.

We shall only farther remark, that the whole Memoir is written with great elegance and perspicuity, and besides being a most remarkable specimen of typographical taste, is indicative of talents of the highest order, combined with great application and deep research. Σ.

ARTICLE XXXVI.

Miscellaneous Notices.

PERIODICALS.

The principal periodicals which have accumulated on the Triangular Tablet of the "New-York Mathematical Club," are:—

1. The American Annals of Education and Instruction, and Journal of Literary Institutions. Published at Boston, in monthly numbers of 40—50 pages 8vo. each. Conducted by William C. Woodbridge, assisted by several friends of Education. Price of subscription, per annum, $3, payable in advance. A work eminently calculated to diffuse useful information throughout America.

2. The American Journal of Arts and Sciences. Published at New-Haven, Con. in quarterly numbers of — pages each. Conducted by Benjamin Silliman, M.D., LL.D. Price of subscription, per annum, $6.

The pages of this useful journal have at times been adorned by the papers of our ingenious and highly esteemed correspondent, Professor T. Strong. They relate principally to the *Principia*, though oc-

casional remarks are elicited in reference to the *Mécaniques Analytique and Céleste*.

They are to be found in volumes 16, 17, 18, 20. Among them we have read with great pleasure new demonstrations of the principle of virtual velocities, and of Kepler's problem; this last is, we believe, entirely original, and in simplicity much excels that of Laplace. We promise our readers a strict analysis of them in our next number.

3. The Annales de Mathematiques Pures et Appliquees. Published at Nismes, France, in monthly 4to. numbers, of 30 pages each. Conducted by M. J. D. Gergonne. Price of subscription, per annum, $6.* We gave some account of this truly excellent and highly creditable work in the last No. of the Diary.

4. The Annals of Philosophy. Published at London, in monthly 8vo. numbers, of 80 pages each. Conducted by Richard Taylor, F.S. A.L.S. &c. and Richard Phillips, F.R.S. L. &c. Price of subscription, per annum, $11,50. This was also noticed in No. XII.

5. The Bulletin des Sciences, Mathematiques, Physiques et Chimiques. Published at Paris, under the general superintendence of the Baron de Ferussac, in monthly 8vo. numbers, of 80 pages each. Conducted by M. Saigey. Price of subscription, per annum, $6. An impartial analyst of all publications and discoveries.

6. The Journal of Mathematics, Pure and Applied. Published at Berlin, under the auspices of the government, in quarterly 4to. numbers, of 112 pages each. Conducted by M. Crelle, Member of the Royal Academy of Sciences, and intimate Counsellor of the King of Prussia. Price of subscription, per annum, $6,50.

7. The Journal of the Royal Institution of Great Britain. Published by John Murray, London, in quarterly 8vo. numbers, of 204 pages each. Price of subscription, per annum, $9. An extremely interesting work.

8. The Quarterly Journal of Education. Published at London, under the superintendence of the Society for the Diffusion of Useful Knowledge, in quarterly 8vo. numbers, of 213 pages each. Price of subscription, per annum, $6. A most judicious and impartial work, stocked with the highest degree of useful information.

9. The Connaissance des Temps ou des Mouvemens Celestes. Published at Paris, for the use of Astronomers and Navigators, by the French Board of Longitude, in an annual 8vo volume of 384 pages, including Additions. Price $1,87½.

Other Interesting Publications.

The Works of F. S. Hassler, F R.S. &c. &c.

1. *A Popular Exposition of the System of the Universe*, 1 vol 8vo. pp. 230, with 5 plates. This book forms an eligible basis for public

* This, and the other prices affixed to foreign publications, are those payable at New-York.

lectures, as well as for collegiate institutions; since, to understand it, requires no previous knowledge of mathematical science, and it also contains all the data to which mathematics *may* be applied in the elucidation of any astronomical phenomena.

2. *Elements of Arithmetic, Theoretical and Practical.* 3d edition (or 2d stereotype.) New-York. G. & C. & H. Carvill. The merits of this useful elementary work, have long been pronounced by the verdict of approbation of both Europe and America.

3. *Elements of the Geometry of Planes and Solids,* 1 vol. 8vo. For sale by G. & C. & H. Carvill. This treatise contains, in 130 propositions, a concise exposition of the principles of this noble science.

4. *Logarithmic and Trigonometric Tables to seven places of Decimals,* in a pocket form, in which all the errors of former tables are corrected. 1 vol. 18mo. New-York, (stereotyped by Hagar, and just published by G. & C. & H. Carvill.) To these excellent Tables is prefixed an introduction of eight pages in five different languages, viz. English, French, German, Spanish, and Latin. They are printed in a peculiarly clear and small type, cast purposely for the occasion; and besides a large number of useful formulæ, contain all the means of accurate calculation which the large 8vo volumes of the French and English (e.g. Callet and Hutton) can furnish.

We are sorry that our limits will not allow us to enter into a closer examination of Mr. Hassler's works; but he is too well known to the world of science, to *need* much comment or praise.

Life of Sir Isaac Newton. By David Brewster, LL.D. (with a portrait.) 1 vol. 12mo. of Harper's Family Library. 1831.—A most interesting and entertaining recreation for the mathematical student, and one of which he will never tire.

Nouvelle Théorie de L'Action Capillaire. Par S. D. Poisson, Membre de L'Institut, du Bureau des Longitudes, &c. &c. &c. 4to. Paris. Bachelier, 1831. pp. 300.

New Theory of Capillary Action. By S. D. Poisson, Member of the Institute, of the Board of Longitude, and of the University of France; of the Royal Societies of London and of Edinburgh; of the Academies of Stockholm, of St. Petersburgh, of Boston, &c. &c.

The well known exceptions to the laws of Hydrostatics, presented to the scientific world in the phenomena of Capillary Action, are in this truly able memoir examined with, and subjected to, the strictest analytical scrutiny, and we can in fact look no where for a more forcible example of the necessity of analytic investigation.

The work of M. Poisson is divided into seven chapters, whose contents are as follows:

1. Preliminary Discussion.
2. Equation of the Capillary Surface.
3. Equation of the Contours of the Capillary Surface.
4. Equilibrium of one or more Liquids in a Capillary Tube.
5. Pressure of Liquids, modified by Capillary Action.
6. Solution of different Problems.
7. Notes and Additions.

ARTICLE XXXVII.

New Questions

TO BE RESOLVED BY CORRESPONDENTS IN NO. XIV.

QUESTION I. (244.)—*By Arithmeticus, New-York.*

If $\frac{n}{D}$ denote any proper fraction, and p, q, r, s, &c. the quotients arising from dividing D by n, D by the remainder of this, and so on; then the fraction $\frac{n}{D}$ may be expressed as follows, $\frac{1}{p} - \frac{1}{pq} + \frac{1}{pqr} - \frac{1}{pqrs}$, &c. Required the demonstration.

QUESTION II. (245)—*By Mr. William Vodges, Philadelphia.*

Given $\left\{ \begin{array}{l} (x^2 + y^2)y = 78 \\ x^4 + y^4 \quad = 97 \end{array} \right\}$ to determine the values of x and y by a quadratic.

QUESTION III. (246.)—*By the same.*

Given $\left\{ \begin{array}{l} x + xy + y \qquad\quad = 34 \\ x^2 + y^2 - (x + y) = 42 \end{array} \right\}$ to determine the values of x and y by a quadratic.

QUESTION IV. (247.)—*By Mr. Edward G. T. Smith, Brooklyn, L. I.*

Given $\left\{ \begin{array}{l} x^3 + xy^2 = ay \\ x^2y + y^3 = bx \end{array} \right\}$ to determine the values of x and y.

QUESTION V. (248.)—*By Mr. Beele, New-York.*

Given $\left\{ \begin{array}{l} x^4 + y = 628 \\ y^4 + z = \ \ 82 \\ z^4 + x = \quad 6 \end{array} \right\}$ to determine the values of x, y, and z.

QUESTION VI. (249.—*By Mr. Michael Floy, Junr. New-York.*

Given $\left\{ \begin{array}{l} xy = z \\ yz = v \\ xv = m \\ yv = px \end{array} \right\}$ to determine the values of x, y, z, and v.

QUESTION VII. (250.)—*By Mr. John D. Williams, New-York.*

It is required to find two triangular numbers such, that their sum and difference shall be hexagonal numbers.

QUESTION VIII. (251.)—*By Mr. C. Gill, Sawpits, New-York.*

It is required to find three *positive* integers such, that their sum and the sum of every two of them may be complete cubes.

QUESTION IX. (252.) —*By Mr. William Lenhart, York, Penn.*

Divide unity into three such positive parts, that if each part be increased by unity, the sums shall be three rational cubes.

NOTE.—This question was originally proposed by Mr. Joseph Waters, Graves Lane, England, in 1817, and fifty pounds sterling offered for a solution. It was also proposed in this country by Mr. Marrat, in his Scientific Journal for April, 1818; but no solution having ever yet appeared, the question is therefore offered again.

QUESTION X. (253.)—*By Mr. P. E. Miles, Buffalo, New-York.*

A gentleman is desirous to lay out ten acres of land in the form of a crescent, bounded by a quadrant and semi-circle; required the diameter of the semi-circle.

QUESTION XI. (254.)—*By Mr. P. Barton, Junr. Schenectady, N. Y.*

It is required to circumscribe the least isosceles triangle about two circles, touching each other, and the base of the triangle; the diameters of the circles being 16 and 20.

QUESTION XII. (255.)—*By Mr. William Lenhart, York, Penn.*

It is required to find a point, P, from which if straight lines be drawn to four other points, A, B, C, D, given in position

$$n \times (PA)^2 + m \times (PB)^2 + r \times (PC)^2 + s \times (PD)^2$$

shall be a minimum.

QUESTION XIII. (256.)—*By X. Y. Z., Brooklyn, Long Island.*

Suppose the frustum of a right cone to be cut by a plane through the contrary extremities of its two diameters; the diagonal of the greater base is nine inches, and the whole surface of the greater ungula (or hoof) is to that of the lesser : : 42 : 23; required the dimensions of the frustum when their simple surfaces are in the same ratio.

QUESTION XIV. (257.)—*By Mr. J. S. Van de Graaff, Lexington, Ken.*

To find on the surface of a given sphere, the area of the greatest triangle, whose perimeter is a semi-circle, and whose greater angle is just double the smaller.

QUESTION XV. (258.)—*By Analyticus, New-York.*

In a plane triangle, having given the base and the rectangle of the sides, to find the locus of the vertex.

QUESTION XVI. (259.)—*By Mr. C. Gill, Sawpits, New-York*

Let tangents be drawn to the curve, as in Question (242) Diary, and perpendiculars let fall upon them from the given point A; it is required to find the locus of their right angles, its caustic by reflection, (the focus of incident rays being at A,) construction, area, length, and the superficies, and solidity of the solid generated by its revolution round XY.

QUESTION XVII. (260.)—*By Mr. Samuel Ward, 3d, New-York.*

It is required to find the equation, &c. of the curve generated by the centre of a circle, rolling upon the evolute of a parabola.

QUESTION XVIII. (261.)—*By Q.*

Required the curve, whose subtangent varies as the *n*th power of the radius of curvature.

QUESTION XIX. (262.)—*By Professor Thomson, Nashville, Tenn.*

Required the length of the involute of the curve which forms the involute of the circle.

QUESTION XX. (263.)—*By Mr. Benjamin Peirce, Cambridge, Mass.*

It is required to prove that the ellipse is the only curve in which the light proceeding from one point is reflected to another point.

QUESTION XXI. (264.)—*By Omicron, North Carolina.*

The latitude of Chapel Hill is 35°, 57′, 21″, N. Now if the rotation of the earth, considered as an ellipsoid should cease, would the weight of a man who now weighs twelve stone then be altered? If so, how much?

QUESTION XXII. (265.)—*By Monge, West Point, N. Y.*

Let A, B, C, D, denote the four angular points of a given parallelogram; it is required to find the inclination of its plane and position of its sides, in order that its projection upon an horizontal plane may be a rhombus, whose opposite angles are 60° and 120° respectively.

QUESTION XXIII. (266.)—*By L'Inconnu, Cincinnati, Ohio.*

It is required to construct the least sphere that shall be a tangent to four given spheres.

QUESTION XXIV. (267.)—*By Mr. George Gregory, England.*

It is required to cut off the frustum of a cone standing upon an horizontal plane, *a part* by a plane passing through the extremity of its upper end, and continued to the base, so that the remaining solid may just support itself from falling; the altitude of the frustum being 8, and the radii of its two ends 2 and 3 inches.

QUESTION XXV. (268.)—*By L'Inconnu, Cincinnati, Ohio.*

It is required to determine all the oscillations that can be made by a thin circular plate, supported by its centre upon a fixed hemisphere; all the particles of the plate being subjected to the action of gravity.

QUESTION XXVI. (269.) or PRIZE QUESTION.—*By Scientific Sigma, Esq. New-York.*

It is required to describe upon the same plane, three circles touching each other, each of which shall touch two given circles.

For the best solution to this question, a handsomely bound complete set of the Diary in two volumes, is offered.

The Editor has also been requested to propose for the consideration of the scientific world, the following:—

QUESTION.—*By Mr. Samuel Ward,* 3d, *New-York.*

It is required to determine all the oscillations of a hollow hemi-ellipsoidal cup, and a given quantity of homogenous fluid contained in it, when the cup rocks without sliding about the shorter axis, upon a fixed horizontal plane.

For a complete solution to the above, the proposer offers a Prize of *one hundred dollars.*

ARTICLE XXXVIII.

ALPHABETICAL INDEX

Of all Contributors to, and Correspondents of, the Mathematical Diary, Nos. I. to XIII, inclusive.

(A.)

Names.	Places of Residence.
Abbott, Robert	*New-York.*
Academicus,	*do.*
Adrain, Robert*, LL.D.	*Prof. of Math. in the University of Penn.*
Aikin, Daniel D.	*Quaker Hill, Dutchess Co. N. Y.*
Aliquis,	*Herkimer, N. Y.*
Alsop, George	*Philadelphia, Penn.*
Analyticus,	*New-York.*
————,	*New-Jersey.*
Anderson, Henry James, M.D.	*Prof. of Math. and Astron. in Col. Coll. New-York.*
Arithmeticus,	*Boston, Mass.*
————,	*Montreal, U. C.*
————,	*New-York.*
Avary, Charles	*Herkimer, N. Y.*

(B.)

Names.	Places of Residence.
Barton, Jacob	
———, P. Junr.	*Schenectady, N. Y.*
Beel, Mr.	*New-York.*
Benedict, Farrand N.	*Montezuma, N. Y.*
Bixby, Alpheus	*New-York.*
Bogert, David S.	*do.*
Bond, Mary	*Frederick, Md.*
Bowditch, Nathaniel, L.LD.	*Boston, Mass.*
———, J. Ingersoll	*do.*
Brady, Thomas S.	*New-York.*
Brown, Matthew, Junr.	*Willingham, Vermont.*
Burger, Devoor V.	*Long Island, N. Y.*
Byrne, Patrick	*Philadelphia, Penn.*

(C.)

Names.	Places of Residence.
Calcul,	*New-York.*
Capp,† John	*Harrisburgh, Penn.*
Carlin, Patrick	*Brooklyn, L. I.*
Carmody, Dennis W.	*New-York.*
Cartesius,	*Cincinnati, Ohio.*
Catlin, Marcus	*Mount Holly, N. J.*
Clowes, Rev. Timothy, LL.D.	*Hempstead, L. I.*
Collins, Matthew	*Prof. of Math. Limerick, Ireland.*
Correspondent, A	*Boston, Mass.*
————————	*Lexington, Ky.*
————————	*New-York.*
Crosby, H.	*Hyattstown, Md.*

* Those gentlemen whose names are in italics have obtained Prizes.

† This esteemed correspondent died February 14th, 1826.

(D.)

Darnall, Henry	*Philadelphia, Penn.*
Davis, Cornelius	*Dutchess County, N. Y.*
Dean, ——	*Prof. in Burlington College, Vermont.*
Delafield, John, Junr.	*New-York.*
Denny, William S.	*Wilmington, Delaware.*
Denovan, James	*New-York.*
Devoy, Edward	*do.*
Diarius,	*do.*
Diophantus,	*Frederick, Maryland.*
————,	*New-York.*
Disney, D. T.	*Cincinnati, Ohio.*
Divver, James	*Prof. in South Carolina Coll. Columbia.*
Docharty, James	*Teacher of Math. Flushing, L. I.*
————, Gerardus B.	*do.*
Donnelly, Nicholas	*New-York.*
Doyle, Rev. Henry	*do.*
————, Michael	*Washington College, Chestertown, Md.*
D. YY.	*Newport, Rhode Island.*

(E.)

Eboracensis,	*New-York.*
Elder, James	*Cincinnati, Ohio.*
Evans, Edward	*Philadelphia, Penn.*
————, George	*New-York.*

(F.)

Farquhar, Charles	*Alexandria, Maryland.*
Farrell, James O.	*New-York.*
Fleming, Peter	*Civil Engineer, New-York.*
Floy, Michael, Junr. A.M.	*New-York.*
Forrest, William	*Mathematical Teacher, New-York.*
Foster, James	*Belfast, Ireland.*
Furber, Frederick	*Graduate of Har. Uni. Cambridge, Mass.*

(G.)

Giddens, Edward	*Fort Niagara, N. Y.*
Gill, C.	*Teacher of Mathematics, Sawpits, N. Y.*
Gregory, George	*England.*
————, Robert	*Nantucket.*

(H.)

Hallowell, Benjamin	*Alexandria, D. C.*
Hamilton, James	*Trenton, New-Jersey.*
Hammond, Selah	*Brooklyn, L. I.*
Hanson, Philip W.	*New-York.*
————, W.	*Lawrenceville, New-Jersey.*
Hussey, B.	
Hypathia,	*Pennsylvania.*

(J.)

James, John F.	*Trenton, New-Jersey.*
Jenkins, John F.	*Middleton Academy, L. I.*
Jones, J. C.	*Bucks County, Penn.*
Juvenis,	*New-York.*

(K.)

Kean, ——	*New-York.*
Keenan, Dennis	*do.*
Kells, William	*Bergen, New-Jersey.*
Knight, Dubre	*Alexandria, D. C.*

(L.)

Lansing, Enoch	*Morrisville, Bucks County, Penn.*
Laws, Saxegotha	*Wilmington, Delaware.*
Lee,* Patrick	*New-York.*
Leeds, N.	*Philadelphia, Penn.*
Lenhart, William	*York, Penn.*
Leonard, Dennis	*New-York.*
Lewis, William I.	*Philadelphia, Penn.*
L'Inconnu,	*Cincinnati, Ohio.*
Loomis, E.	*Baltimore, Maryland.*
Lott, Henry R.	*New-York.*
Lynch, Elias	*do.*

(M.)

Macully, James	*Richmond, Virginia.*
Mathetus,	*Bucks County, Penn.*
Matthews, Thomas J.	*Prof. of Math. in Trans. Coll. Lexington, Kentucky.*
M'Gowan, B.	*Teacher of Mathematics, New-York.*
M'Ginness, James	*Harrisburgh, Penn.*
M'Keen, Joseph	*New-York.*
M'Vickar, Samuel Bard	*do.*
Megear, Thomas J.	*Wilmington, Delaware.*
Member of the Math. Club,	*New-York.*
Miles, P. E.	*Buffalo, New-York.*
Monge,	*West Point, N. Y.*
Mooney, Thomas, Junr.	*Brooklyn, L. I.*
Moreau, John B.	*New-York.*

(N.)

Nash, M.	*New-York.*
Nassau,	*Long Island.*
Nemo,	*New-York.*
——,	*Ohio.*
Newman, John B.	*New-York.*
Nichols, Walter	*do.*
———, W.	*Edinburgh, Scotland.*
N'Importe Qui,	*New-York.*
Noll, Frederick	*Harlaem, New-York.*

* This estimable mathematician died in the fall of 1830.

Nulty, Eugenius	*Philadelphia, Penn.*
Numskull, Old	*New-York.*
	(O.)
Omicron,	*North Carolina.*
O'Shannessy, M., A.M.	*Professor of Mathematics, New-York.*
O'Sullivan, John	*U. S. Navy Yard, Pensacola, Florida.*
Otis, Charles P.	*Colchester, Mass.*
	(P.)
Paine, J. Leslie	*Halifax, Nova Scotia.*
Parry, Robert	*Vincent-town, New-Jersey.*
Pascalis, Cyril O.	*New-York.*
Patterson, Walter	*do.*
Peirce, Benjamin	*Math. Instructor at Harvard University, Cambridge, Mass.*
Phillips, James	*Prof. of Math. Chapel Hill Coll. N. C.*
Philomath,	*Albany, New-York.*
———,	*Winchester, Virginia.*
Philotechnus,	*Philadelphia, Penn.*
Plus Minus,	*New-York.*
Potts, Charles	*Philadelphia, Penn.*
Purcell, William	*Brooklyn, L. I.*
	(Q.)
Quin, James	*New-York.*
Quinby, A. B.	*do.*
Q. U. Z.	*Oxford, Ohio.*
Q.	
	(R.)
Reid, John C.	*New-York.*
Roche, Martin	*Philadelphia, Penn.*
Rochford,* John	*New-York.*
Rockwell, E. H.	*Frederick, Maryland.*
Rodriguez, J. P.	*Gosport, Virginia.*
Root, O.	*Vernon, New-York.*
	(S.)
Scally, William	*New-York.*
Senex,	*do.*
Shanley, Daniel	*Charleston, South Carolina.*
Sherry,† Francis	*New-York.*
Shirer, Augustus	*Northampton, Mass.*
Sidell, William H.	*Cadet U. S. Mil. Acad. Westpoint, N. Y.*
Sigma, Scientific, Esq.	*New-York.*
Sloane, James	*New-York.*
Smith, Augustus W.	
———, Edward G T.	*Brooklyn, L. I.*
———, John	*Cincinnati, Ohio.*
———, Seth	*Philadelphia, Penn.*
———, William S.	*Natchez, Mississippi.*

* We announce with regret. the premature death of this gentleman—he died in Virginia.

† For the particulars of the decease of this ingenious gentleman, see p. 224.

Stigleman, Wm. A. W.	*Harrisburgh, Penn.*
Strode, Joseph C.	*Strodesville, Penn.*
Strong, Theodore	*Prof. of Mathematics, Rutgers College, New-Brunswick, N. J.*
Student, A, of Col. Coll.	*New-York.*
———, of Rutgers Coll.	*New-Brunswick, New-Jersey.*
Swale, J. H.	*Liverpool, England.*
Sweeny, James	*New-York.*
Swinburne, John	*Brooklyn, L. I.*
S. Y.	*Of Albany, New-York.*
S——,	*Of G.*
S——,	*Of Brooklyn, L. I.*
S——,	*Of New-York.*
	(T.)
Thomson, S.	*Prof. of Math. in Nashville Univ. Tenn.*
———, William	*Saugerties, New-York.*
Trigonometricus,	*Alabama.*
Tyro,	*Brooklyn, L. I.*
——,	*Lexington, Kentucky.*
	(V.)
Van de Graaff, J. S.	*Lexington, Kentucky.*
Vernon, N.	*Fredericktown, Maryland.*
Vodges, William	*Philadelphia, Penn.*
Vose, Henry	*Mem. of the Louisiana Bar, N. Orleans.*
Vyse, Charles	
	(W.)
Wallace, Patterson	*New-York.*
Walker, Sears C.	*Philadelphia, Penn.*
Ward, Edward C.	*New-York.*
——, Farrell	*do.*
——, Samuel, 3d	*Graduate of Col. Coll. New-York.*
Warner, Silas	*Wrightstown, Penn.*
Warnock, James	*New-York.*
Waterman, Barclay	*Philadelphia, Penn.*
Wells, Ransford	*New-Brunswick, New-Jersey.*
Wiggins, Benjamin	*Bucks County, Penn.*
Wighton, Charles	
Wilder, Cephas	*Baltimore, Maryland.*
Willetts, Jesse	*Maiden Creek, Bucks Co. Penn.*
———, John H.	
Williams, John D.	*New-York.*
Wilt, John M.	*Springfield, Penn.*
Wright, Solomon	*Lumberville, Bucks Co. Penn.*
	(X.)
XYZ	*Brooklyn, L. I.*
——	*New-York.*
X Y.	*Charleston, South Carolina.*

END OF VOL. II.